# Exact Markovian kinetic equation for a quantum Brownian oscillator

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We derive an exact Markovian kinetic equation for an oscillator linearly coupled to a heat bath, describing quantum Brownian motion. Our work is based on the subdynamics formulation developed by Prigogine and collaborators. The space of distribution functions is decomposed into independent subspaces that remain invariant under Liouville dynamics. For integrable systems in Poincaré's sense the invariant subspaces follow the dynamics of uncoupled, renormalized particles. In contrast, for nonintegrable systems, the invariant subspaces follow a dynamics with broken time symmetry, involving generalized functions. This result indicates that irreversibility and stochasticity are exact properties of dynamics in generalized function spaces. We comment on the relation between our Markovian kinetic equation and the Hu-Paz-Zhang equation.

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## I. INTRODUCTION

A well-known model of quantum Brownian motion is a harmonic oscillator linearly coupled to a bath of field modes. The Hamiltonian is (with $\hbar=1$)

$$H=\frac{1}{2M_1}p_1^2+\frac{m_1}{2}\omega_1^2q_1^2+\sum_{k=0}^{\infty}\frac{1}{2M_k}p_k^2+\frac{m_k}{2}\omega_k^2q_k^2+\lambda\sum_{k=0}^{\infty}C_kq_1q_k, \tag{1}$$

where $q_1$ and $p_1$ are the positions and momenta of the harmonic oscillator and $q_k$ and $p_k$ are positions and momenta of the field oscillators. Here $k$ are the wave numbers, and $\lambda$ is a dimensionless coupling constant.[1]

The Hamiltonian (1) has been considered in numerous papers (see Dekker's review [1]). Hu, Paz, and Zhang have obtained an exact equation for the reduced density matrix of the oscillator using a path-integral method,

$$\begin{aligned}i\frac{\partial}{\partial t}\rho_r=&\left[-\frac{\omega_1}{2}\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial {x'}_1^2}\right)+\frac{\tilde{\Omega}^2(t)}{2\omega_1}(x_1^2-{x'}_1^2)\right]\rho_r-i\,\Gamma(t)\\&\times(x_1-x_1')\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x'_1}\right)\rho_r-\frac{i}{\omega_1}\Gamma(t)h(t)(x_1-x_1')^2\rho_r\\&+\Gamma(t)f(t)(x_1-x_1')\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x'_1}\right)\rho_r,\end{aligned} \tag{2}$$

where

$$x_1=\sqrt{M_1\omega_1}q_1,\quad x'_1=\sqrt{M_1\omega_1}q'_1 \tag{3}$$

are dimensionless coordinates, and the time-dependent coefficients are defined in Ref. [2].

The terms with $\Gamma(t)$ and $\Gamma(t)h(t)$ on the right hand side of Eq. (2) suggest the existence of damping and diffusion processes characteristic of Brownian motion. Strictly speaking though, the Hu-Paz-Zhang (HPZ) equation (2) is time-reversal invariant and it corresponds to a deterministic evolution of wave functions. Indeed, the solution of the HPZ equation is equivalent to the solution obtained from Schrödinger's equation, i.e.,

$$\rho_r(t)=\mathrm{Tr_F}[e^{-iHt}\rho(0)e^{iHt}] \tag{4}$$

where $\mathrm{Tr_F}$ means the trace over the field and

$$\rho(0)=\sum_\alpha\rho_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| \tag{5}$$

is the initial density matrix, diagonalized in a suitable basis of wave functions $|\psi_\alpha\rangle$. The wave functions $|\psi_\alpha\rangle$ form a complete orthonormal basis of the whole system of harmonic oscillator and heat bath. Due to the equivalence (4), the HPZ equation describes a time-reversible, deterministic evolution of each wave function. This contrasts with true Brownian motion, described by a Markovian equation with broken time symmetry, which corresponds to a stochastic evolution of each wave function $|\psi_\alpha\rangle$ [3].

The derivation of irreversible Markovian equations from dynamics has been a great challenge [4]. This is related to the apparent incompatibility between the second law of thermodynamics and time-reversible dynamics. One point of view is that Markovian equations appear as an approximation of the dynamical equations. This is the so-called Markovian approximation, valid for weak coupling between interacting particles, and for time scales of the order of the relaxation time to equilibrium [5].

However, one can take a different point of view, where Markovian equations are formally derived from dynamical equations without any approximation. This is the *subdynamics* formulation developed by Prigogine and collaborators

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[1]Note that for a finite system the wave numbers $k$ are not integers [see Eq. (6)]. Thus the bath variables (e.g., $q_k$ and $p_k$) do not take the "1" index reserved for the harmonic oscillator variables $q_1$ and $p_1$.

 

[4,6–11]. In this approach, essential elements are the distinction between integrable and nonintegrable systems in the sense of Poincaré, and the use of generalized functions [12]. In this paper we derive an exact Markovian equation for the quantum Brownian oscillator, based on this approach. This equation is valid for both weak and strong coupling. As we will show, for weak coupling, it agrees with the HPZ equation.

A few exact results using subdynamics have already been obtained [4,13]. However, to our knowledge, there was no derivation of an exact Markovian equation for quantum Brownian motion. Previous formulations were centered on density operators. Here we focus on the observables, that is, on products of creation and annihilation operators. This allows us to consider arbitrary $N$-particle sectors in a nonperturbative way.

This paper is organized as follows. In Sec. II we introduce our formulation of subdynamics. As in the original formulation, we introduce the projection superoperators $\Pi$ and $\hat{\Pi}=1-\Pi$, which define invariant subspaces of the Liouville superoperator. In Sec. III we define integrability and nonintegrability in Poincaré's sense. In Secs. IV and V, we construct the $\Pi$ projector for the integrable case and derive recursive relations, for this projector. Extending these relations, we construct $\Pi$ for the nonintegrable case in Sec. VI. This leads to our Markovian equation in Sec. VII. In Sec. VIII we compare this equation with the HPZ equation. Concluding remarks are presented in Sec. IX. Additional calculations are presented in the Appendixes.

## II. SUBDYNAMICS

In this section we introduce the main ideas of our approach. We focus on the quantum Brownian oscillator model.

We consider a one-dimensional space. We start with the system in a box of size $L$ and impose periodic boundary conditions. Then in Eq. (1) we have

$$k=2\pi n/L \tag{6}$$

with integer $n$. We are interested in the limit $L\to\infty$, where the spectrum of field frequencies $\omega_k$ becomes continuous. We will assume that $C_k=C_{-k}$ and $\omega_k=\omega_{-k}$. This allows us to restrict $k\geqslant 0$, keeping only the symmetric part of the $q_k$ operators, i.e., we set $q_k=q_{-k}$. We will assume as well that there is no degeneracy in the spectrum of $\omega_k$ for $k\geqslant 0$.

It will be convenient to express the Hamiltonian (1) in terms of annihilation and creation operators. We express the coordinates $q_i$ as

$$q_i=\frac{1}{\sqrt{2M_i\omega_i}}(a_i+a_i^\dagger), \quad i=1,k, \tag{7}$$

where $a_i^\dagger$ and $a_i$ are bosonic creation and annihilation operators of the particle $(i=1)$ and field $(\{i\}=\{k\})$. These operators satisfy the usual commutation relations

$$[a_i,a_j^\dagger]=\delta_{ij}, \tag{8}$$

for $i$, $j=1$ or $k$. For the momenta we have

$$p_i=-i\sqrt{\frac{M_i\omega_i}{2}}(a_i-a_i^\dagger), \quad i=1,k. \tag{9}$$

Introducing the notation

$$V_k=C_k/\sqrt{4M_1\omega_1M_k\omega_k}, \tag{10}$$

the Hamiltonian takes the form [14,15]

$$H=\omega_1a_1^\dagger a_1+\sum_{k=0}^{\infty}\omega_ka_k^\dagger a_k+\lambda\sum_{k=0}^{\infty}V_k(a_1^\dagger+a_1)(a_k^\dagger+a_k)+E_{\text{vac}}, \tag{11}$$

where $E_{\text{vac}}$ is the vacuum energy. The interaction has the following volume dependence:

$$V_k=\left(\frac{2\pi}{L}\right)^{1/2}v_k, \tag{12}$$

with $v_k$ independent of $L$. In the limit $L\to\infty$, the sum over discretized field modes turns into an integral and the Kronecker $\delta$ function turns into a Dirac $\delta$ function,

$$\frac{2\pi}{L}\sum_k\to\int dk, \quad \frac{L}{2\pi}\delta_{k,k'}\to\delta(k-k'). \tag{13}$$

Hereafter, whenever we write summations or Kronecker $\delta$ functions, it is understood that we take the limit $L\to\infty$ using Eq. (13). Also from now on, when we take the limit $L\to\infty$ we will keep the energy density of the field finite. This means that [with $\langle A\rangle=\text{Tr}(A\rho)$]

$$\langle a_k^\dagger a_k\rangle\sim L^0 \quad \text{for } L\to\infty. \tag{14}$$

This condition is known as the thermodynamic limit. Moreover, we consider density operators that have diagonal ($\delta$-function) singularities in field-mode representation [10,16]. An example of this class of ensembles is the equilibrium Gibbs distribution. For these density operators,

$$\sum_{k'}\langle a_k^\dagger a_{k'}\rangle\sim\langle a_k^\dagger a_k\rangle\sim O(L^0). \tag{15}$$

Diagonal observables are as important as sums of off-diagonal observables. Due to this property the separation of diagonal and off-diagonal observables, which we consider below, is well defined in the thermodynamic limit.[2]

The Hamiltonian in Eq. (11) has the form

$$H=H_0+\lambda V \tag{16}$$

where $H_0$ is the unperturbed part describing free motion, and $V$ is the interacting part. Corresponding to this Hamiltonian we have the Liouville superoperator (or "Liouvillian")

$$L_H=[H,\ ]=L_0+\lambda L_V. \tag{17}$$

From the Liouville equation

[2]It is possible to avoid summations altogether, and use integrals from the beginning [17]. The results are the same.

$$i\frac{\partial}{\partial t}\rho(t) = L_H\rho(t), \tag{18}$$

we obtain the time evolution of averages of observables $W$,

$$\langle W(t)\rangle = \mathrm{Tr}[W\rho(t)]. \tag{19}$$

We will consider observables depending only on particle operators, expandable in monomials,

$$W = a_1^{\dagger n} a_1^m, \tag{20}$$

with $m$, $n \geqslant 0$ integers. Then we have

$$\langle W(t)\rangle = \mathrm{Tr}[WP\rho(t)] \tag{21}$$

where $P$ is a linear projection superoperator defined by

$$P\left(a_1^{\dagger m_1} a_1^{n_1} \prod_{k=0}^{\infty} a_k^{\dagger m_k} a_k^{n_k}\right) = a_1^{\dagger m_1} a_1^{n_1} \prod_k \delta_{m_k,n_k} a_k^{\dagger m_k} a_k^{m_k}, \tag{22}$$

for $m_i$, $n_i \geqslant 0$. This projector singles out products of creation and annihilation operators with diagonal field operators. Every creation operator $a_k^\dagger$ present in the product has to be paired with the annihilation operator $a_k$.

The projector $P$ commutes with the free Liouvillian $L_0$,

$$PL_0 = L_0 P. \tag{23}$$

Introducing the complementary projector $Q = 1 - P$ we have $PQ = QP = 0$. Thus under the unperturbed time evolution (with $\lambda = 0$), any density operator can be decomposed into two components that evolve independently,

$$\rho = P\rho + Q\rho. \tag{24}$$

Each component remains invariant under the free time evolution. For $\lambda = 0$ we have

$$i\frac{\partial}{\partial t}P\rho(t) = L_0 P\rho(t),$$

$$i\frac{\partial}{\partial t}Q\rho(t) = L_0 Q\rho(t). \tag{25}$$

Each component follows its own subdynamics, with closed time evolution. This separation allows us to calculate $\langle W(t)\rangle$ knowing only the $P\rho$ component of $\rho$, without the complementary $Q\rho$ component.

On the other hand, the interacting Liouvillian (with $\lambda \neq 0$) does not commute with $P$. We have

$$i\frac{\partial}{\partial t}P\rho(t) = PL_H\rho(t) = PL_H P\rho(t) + PL_H Q\rho(t). \tag{26}$$

This is no longer a closed equation for the component $P\rho(t)$. This is the main problem of nonequilibrium statistical mechanics. A common approach to deal with this problem is to write a hierarchy of equations of the Bogoliubov-Born-Green-Kirkwood-Yvon type [4]. Alternatively, one can try to obtain closed non-Markovian equations (with memory terms), such as the Prigogine-Resibois generalized master equation [18]. As shown by Hu, Paz, and Zhang, for the quantum Brownian oscillator it is indeed possible to obtain the closed non-Markovian equation (2) for the reduced density matrix. The non-Markovian character of the equation is manifested in the time-dependent coefficients.

In the subdynamics approach, we introduce a projector $\Pi$ satisfying the following three conditions:

$$\text{(A)} \quad \Pi^2 = \Pi,$$

$$\text{(B)} \quad \Pi L_H = L_H \Pi,$$

$$\text{(C)} \quad \Pi = P + \lambda\Pi_1 + \lambda^2\Pi_2 + \cdots,$$

where $\Pi_n$ are independent of $\lambda$. The last condition means that $\lim_{\lambda\to 0}\Pi = P$ and $\Pi$ is analytic at $\lambda = 0$.[3]

Using $\Pi$ we can decompose a density operator into two components that evolve independently:

$$\rho = \Pi\rho + \hat{\Pi}\rho \tag{27}$$

where $\hat{\Pi} = 1 - \Pi$. Each component obeys a closed equation

$$i\frac{\partial}{\partial t}\Pi\rho = L_H\Pi\rho, \tag{28}$$

$$i\frac{\partial}{\partial t}\hat{\Pi}\rho = L_H\hat{\Pi}\rho. \tag{29}$$

Hereafter we will focus on the $\Pi$ component. As we will see, this component gives the closed Markovian equation describing quantum Brownian motion. The complementary component $\hat{\Pi}$ gives memory effects associated with dressing [19].[4]

We focus on the equation

$$i\frac{\partial}{\partial t}\mathrm{Tr}(W\Pi\rho) = \mathrm{Tr}(WL_H\Pi\rho). \tag{30}$$

In the following, we will derive an explicit form of this equation. Using the property

$$\mathrm{Tr}(A \cdot S\rho) = \mathrm{Tr}[(S^\dagger A^\dagger)^\dagger \rho], \tag{31}$$

where $S$ is a superoperator and $A$ is an arbitrary operator, we have

$$i\frac{\partial}{\partial t}\mathrm{Tr}(W\Pi\rho) = \mathrm{Tr}[(L_H\Pi^\dagger W^\dagger)^\dagger \rho]. \tag{32}$$

To obtain the kinetic equation, we need to calculate the quantity $L_H\Pi^\dagger W^\dagger$ with $W = a_1^{\dagger n} a_1^m$. This will be done in the following sections.

We note that [see Eq. (21)]

[3]As shown in Ref. [4] condition (A) actually follows from conditions (B) and (C). In this paper we will verify that all three conditions are satisfied.

[4]In a more detailed formulation of subdynamics (see Ref. [11]) both $\Pi$ and $\hat{\Pi}$ are further decomposed into a sum of orthogonal projectors $\Pi = \Sigma_\nu \Pi^{(\nu)}$ and $\hat{\Pi} = \Sigma_\nu \hat{\Pi}^{(\nu)}$. Each subcomponent gives a closed Markovian equation. However, sums of these projectors can give a non-Markovian equation, as is the case for $\hat{\Pi}$.

$$\mathrm{Tr}(WL_H\Pi\rho) = \mathrm{Tr}(WPL_H\Pi\rho). \tag{33}$$

As shown in Refs. [4,10], we have

$$PL_H\Pi\rho = \theta\Pi\rho, \tag{34}$$

where $\theta$ is a "collision" superoperator satisfying the relation $[\theta, P]=0$. We get an exact, closed Markovian equation

$$i\frac{\partial}{\partial t}P\Pi\rho = \theta P\Pi\rho \tag{35}$$

for the component $P\Pi\rho$. This will be verified through the direct calculation of Eq. (30).

In the construction of $\Pi$ we will consider two cases discussed next: the integrable and nonintegrable cases. As we will show, the Markovian dynamics of Brownian motion occurs in the nonintegrable case. Our approach will be to first obtain $\Pi$ for the integrable case, and then extend this result to the nonintegrable case.

## III. INTEGRABLE AND NONINTEGRABLE CASES

In this section we specify what we mean by integrable and nonintegrable cases.

For the integrable case, the $P$ and $\Pi$ projectors can be related by a similitude transformation

$$\Pi = U^{-1}PU \tag{36}$$

where $U$ is a *time-independent* unitary transformation. This is the same transformation that puts the Hamiltonian in a diagonal form with no interactions [see Eqs. (42) and (43) below]. In this way the interacting system can be mapped to a noninteracting system through a unitary transformation. We call this case "integrable" because there exists a one-to-one correspondence between unperturbed and perturbed invariants of motion. Furthermore, the perturbed invariants are expandable around $\lambda=0$. These properties were studied by Poincaré in the context of celestial mechanics, so when we speak about integrability, it is in Poincaré's sense [20].

In contrast, for the nonintegrable case the interactions cannot be transformed away through a unitary transformation. There is no longer a one-to-one correspondence between unperturbed and perturbed invariants. The $P$ and $\Pi$ projectors are now related by a nonunitary transformation $\Lambda$,

$$\Pi = \Lambda^{-1}P\Lambda. \tag{37}$$

As shown in Refs. [21,22] the transformation $\Lambda$ is "star unitary." In this paper we will construct the $\Pi$ projector directly, without using the $\Lambda$ transformation. Let us just make a few remarks on this transformation. Rather than transforming away the interactions, $\Lambda$ takes us from the original representation in terms of bare particles to a new representation in terms of dressed particles which obey stochastic equations breaking time symmetry. In this representation the effects of noise appear due to the nondistributive character of $\Lambda$ with respect to multiplication [21–24].

For the quantum Brownian oscillator we can have both integrable and nonintegrable cases, depending on the relation between the frequency of the particle and the frequencies of the field modes.

We assume that the field frequencies $\omega_k$ take the values

$$0 \leqslant \omega_0 \leqslant \omega_k < \infty. \tag{38}$$

Here $\omega_0$ is the lower bound of the spectrum of $\omega_k$ for $k=0$.

The integrable and nonintegrable cases correspond, respectively, to the following two possibilities [14]:

$$\text{(a)} \quad \omega_1 < \omega_0, \tag{39}$$

$$\text{(b)} \quad \omega_c < \omega_1, \tag{40}$$

where

$$\omega_c^2 = \omega_0^2 + \int_0^\infty dk \frac{4\omega_c\omega_k\lambda^2 v_k^2}{\omega_k^2 - \omega_0^2}. \tag{41}$$

The frequency $\omega_c$ is a threshold frequency for $\omega_1$, below which the oscillator becomes undamped. The intermediate case $\omega_0 < \omega_1 < \omega_c$ gives undamped oscillations as well. In this case the $\Pi$ and $P$ projectors are related through a unitary transformation, but this transformation is not expandable around $\lambda=0$. This intermediate case will not be considered here. Interesting phenomena associated with this case have been considered in Refs. [25,26].

## IV. Π IN THE INTEGRABLE CASE

We consider now the integrable case (a) discussed in Sec. III. In this case the particle cannot resonate with the field modes. The Hamitonian (11) can be diagonalized through the unitary superoperator $U$ into the following form [27]:

$$H = \bar{\omega}_1\bar{A}_1^\dagger\bar{A}_1 + \sum_{k=0}^{\infty}\omega_k\bar{A}_k^\dagger\bar{A}_k + \bar{E}_{\mathrm{vac}}, \tag{42}$$

where $\bar{\omega}_1$ is the renormalized frequency of the particle, $\bar{E}_{\mathrm{vac}}$ is the renormalized vacuum energy, and the operators $A$ are the renormalized operators replacing the bare operators $a$. The $A$ operators are given by the unitary transformation

$$\bar{A}_i^\dagger = U^{-1}a_i^\dagger,$$

$$\bar{A}_i = U^{-1}a_i. \tag{43}$$

We use overbars to denote variables in the integrable case. The transformed operators satisfy the relations

$$L_H\bar{A}_1^\dagger = \bar{\omega}_1\bar{A}_1^\dagger, \quad L_H\bar{A}_1 = -\bar{\omega}_1\bar{A}_1,$$

$$L_H\bar{A}_k^\dagger = \omega_k\bar{A}_k^\dagger, \quad L_H\bar{A}_k = -\omega_k\bar{A}_k. \tag{44}$$

As mentioned in the previous section, we call this case integrable because this system follows Poincaré's criterion of integrability. There is a one-to-one correspondence between the unperturbed invariants of motion $a_i^\dagger a_i$ and the perturbed invariants $\bar{A}_i^\dagger\bar{A}_i$. The perturbed invariants are expandable around $\lambda=0$.

The superoperator $U^{-1}$ may be written in the form

$$U^{-1}a = u^{-1}au \tag{45}$$

where $u$ is a unitary operator. Thus we have the distributive property

$$U^{-1}ab = [U^{-1}a][U^{-1}b]. \tag{46}$$

It follows that the operators $\bar{A}$ satisfy the commutation relations

$$[\bar{A}_i, \bar{A}_j^\dagger] = \delta_{ij} \tag{47}$$

where $i, j=1, k$.

The transformed operators are given by the linear relations

$$\bar{A}_1^\dagger = \bar{c}_{11}^* a_1^\dagger + \bar{d}_{11}^* a_1 + \sum_k \bar{c}_{1k}^* a_k^\dagger + \sum_k \bar{d}_{1k}^* a_k, \tag{48}$$

$$\bar{A}_k^\dagger = \bar{c}_{k1}^* a_1^\dagger + \bar{d}_{k1}^* a_1 + \sum_{k'} \bar{c}_{kk'}^* a_{k'}^\dagger + \sum_k \bar{d}_{kk'}^* a_{k'}, \tag{49}$$

with the coefficients $\bar{c}$ and $\bar{d}$ written in Appendix A.

The renormalized frequency $\bar{\omega}_1$ is the solution of the equation

$$G(\bar{\omega}_1)^{-1} = 0 \tag{50}$$

satisfying the condition $\lim_{\lambda\to 0}\bar{\omega}_1 = \omega_1$, where $G$ is the Green's function,

$$G(\omega) = \left(\omega_1^2 - \omega^2 - \int_0^\infty dk \frac{4\omega_1\omega_k\lambda^2 v_k^2}{\omega_k^2 - \omega^2}\right)^{-1}, \tag{51}$$

defined here for $\omega < \omega_0$. Using the commutation relations (47) we invert (48) to obtain

$$a_1^\dagger = \bar{c}_{11}\bar{A}_1^\dagger - \bar{d}_{11}^*\bar{A}_1 + \sum_k \bar{c}_{k1}\bar{A}_k^\dagger - \sum_k \bar{d}_{k1}^*\bar{A}_k. \tag{52}$$

We verify now that in the integrable case, the $\bar{\Pi}$ projector is obtained through the relation

$$\bar{\Pi} = \bar{\Pi}^\dagger \equiv U^{-1}PU, \tag{53}$$

where we use an overbar to remind us that this corresponds to the integrable case. To prove this, we will check that this expression satisfies the conditions (A)–(C) in Sec. II.

Condition (A) is satisfied, since $P=P^2$ itself is a projector. Condition (B) means that

$$U^{-1}PUL_H = L_H U^{-1}PU \tag{54}$$

or

$$PUL_H U^{-1} = UL_H U^{-1}P. \tag{55}$$

Using the distributive relation (46) together with Eq. (42) one can show that $UL_HU^{-1}$ has the same form as the unperturbed Liouvillian $L_0$, which implies Eq. (55) is true [22]. Condition (C) is satisfied, since the superoperator $U$ reduces to the unit superoperator when $\lambda \to 0$, as can be seen in the explicit forms of the coefficients in Eq. (48); see Appendix A.

The distributive relation (46) together with Eq. (53) shows that the projector $\bar{\Pi}$ can be defined through the relation

$$\bar{\Pi}\left(\bar{A}_1^{\dagger m_1}\bar{A}_1^{n_1}\prod_{k=0}^{\infty}\bar{A}_k^{\dagger m_k}\bar{A}_k^{n_k}\right) = \bar{A}_1^{\dagger m_1}\bar{A}_1^{n_1}\prod_k \delta_{m_k,n_k}\bar{A}_k^{\dagger m_k}\bar{A}_k^{m_k}, \tag{56}$$

since this is equivalent to Eq. (22).

Note that $\bar{\Pi}^\dagger = \bar{\Pi}$. Henceforth we write $\bar{\Pi}^\dagger$ anticipating the extension to the nonintegrable case [see Eq. (32)].

## V. RECURSIVE RELATIONS

Before going to the nonintegrable case, we will derive recursive relations for the $\bar{\Pi}$ projector in the integrable case. Subsequently, these will be extended to the nonintegrable case as a crucial step in the derivation of our Markovian equation. The relations are

$$\bar{\Pi}^\dagger(a_1^{\dagger m}a_1^n) = \bar{\Pi}^\dagger a_1^\dagger \cdot \bar{\Pi}^\dagger(a_1^{\dagger m-1}a_1^n) + (m-1)\bar{X}\bar{\Pi}^\dagger(a_1^{\dagger m-2}a_1^n) + n\bar{Y}\bar{\Pi}^\dagger(a_1^{\dagger m-1}a_1^{n-1}) \tag{57}$$

and

$$\bar{\Pi}^\dagger(a_1^{\dagger m}a_1^n) = \bar{\Pi}^\dagger(a_1^{\dagger m}a_1^{n-1}) \cdot \bar{\Pi}^\dagger a_1 + (n-1)\bar{X}\bar{\Pi}^\dagger(a_1^{\dagger m}a_1^{n-2}) + m\bar{Y}\bar{\Pi}^\dagger(a_1^{\dagger m-1}a_1^{n-1}), \tag{58}$$

where

$$\bar{X} = -\sum_k \bar{c}_{k1}\bar{d}_{k1}^*\{\bar{A}_k, \bar{A}_k^\dagger\}, \tag{59}$$

$$\bar{Y} = \sum_k |\bar{c}_{k1}|^2\bar{A}_k^\dagger\bar{A}_k + |\bar{d}_{k1}|^2\bar{A}_k\bar{A}_k^\dagger \tag{60}$$

and $\{\}$ are the anticommutator brackets.

In the rest of this section we present the proof of these relations. To facilitate our construction, we define two new operators, i.e.,

$$\bar{B}_1^\dagger \equiv \bar{\Pi}^\dagger a_1^\dagger = \bar{c}_{11}\bar{A}_1^\dagger - \bar{d}_{11}^*\bar{A}_1 \tag{61}$$

[see Eq. (52)] and

$$\bar{D}_k^\dagger \equiv \bar{c}_{k1}\bar{A}_k^\dagger - \bar{d}_{k1}^*\bar{A}_k \tag{62}$$

with their Hermitian conjugates $\bar{B}_1$ and $\bar{D}_k$.

We have

$$a_1^\dagger = \bar{B}_1^\dagger + \sum_k \bar{D}_k^\dagger. \tag{63}$$

Thus

$$\begin{aligned}\bar{\Pi}^\dagger(a_1^{\dagger m}a_1^n) &= \bar{\Pi}^\dagger\left[\left(\bar{B}_1^\dagger + \sum_k \bar{D}_k^\dagger\right)^m\left(\bar{B}_1 + \sum_k \bar{D}_k\right)^n\right] \\ &= \bar{\Pi}^\dagger\left[\bar{B}_1^\dagger\left(\bar{B}_1^\dagger + \sum_k \bar{D}_k^\dagger\right)^{m-1}\left(\bar{B}_1 + \sum_k \bar{D}_k\right)^n\right] \\ &\quad + \bar{\Pi}^\dagger\left[\sum_k \bar{D}_k^\dagger\left(\bar{B}_1^\dagger + \sum_k \bar{D}_k^\dagger\right)^{m-1}\left(\bar{B}_1 + \sum_k \bar{D}_k\right)^n\right].\end{aligned} \tag{64}$$

Using the projection property (56) of $\bar{\Pi}$, the first term of the last expression equals

$$\bar{\Pi}^\dagger\left[\bar{B}_1^\dagger\left(\bar{B}_1^\dagger+\sum_k \bar{D}_k^\dagger\right)^{m-1}\left(\bar{B}_1+\sum_k \bar{D}_k\right)^n\right] = \bar{B}_1^\dagger\cdot\bar{\Pi}^\dagger\left[\left(\bar{B}_1^\dagger+\sum_k \bar{D}_k^\dagger\right)^{m-1}\left(\bar{B}_1+\sum_k \bar{D}_k\right)^n\right] = \bar{\Pi}^\dagger a_1^\dagger\cdot\bar{\Pi}^\dagger(a_1^{\dagger m-1}a_1^n). \quad (65)$$

Now consider the second term on the right hand side of Eq. (64). Expanding this in binomial series we have

$$\bar{\Pi}^\dagger\left[\sum_k \bar{D}_k^\dagger\cdot\left(\bar{B}_1^\dagger+\sum_k \bar{D}_k^\dagger\right)^{m-1}\left(\bar{B}_1+\sum_k \bar{D}_k\right)^n\right] = \bar{\Pi}^\dagger\left[\sum_k \bar{D}_k^\dagger\cdot\sum_{l=0}^{m-1}C_l^{m-1}\bar{B}_1^{\dagger m-1-l}\left(\sum_{k'}\bar{D}_{k'}^\dagger\right)^l \times\sum_{l'=0}^{n}C_{l'}^n\bar{B}_1^{n-l'}\left(\sum_{k''}\bar{D}_{k''}\right)^{l'}\right] \quad (66)$$

where $C_l^m=m!/[(m-l)!l!]$. We have shifted the $\bar{D}$ freely among $\bar{B}$ since the $\bar{D}$ operators commute with the $\bar{B}$ operators [see Eq. (47)].

Due to $\bar{\Pi}^\dagger$, in order to produce a nontrivial projection the term $\Sigma_k\bar{D}_k^\dagger$ at the beginning of the product must pair up with one of the $\Sigma_{k'}\bar{D}_{k'}^\dagger$ with $l$ possible ways or one of the $\Sigma_{k''}\bar{D}_{k''}$ with $l'$ possible ways. Checking the volume dependence, we can neglect simultaneous pairings of $\Sigma_k\bar{D}_k^\dagger$ with three or more $\bar{D}_k$ or $\bar{D}_k^\dagger$, because such terms will be of order $O(1/L)$ in the thermodynamic limit, and therefore can be dropped in comparison with other more dominant terms. For example,

$$\bar{\Pi}^\dagger\left[\sum_k(\bar{D}_k^\dagger)^2\left(\sum_{k'}\bar{D}_{k'}^\dagger\right)^2\right]=\sum_k|\bar{c}_{k1}|^4\bar{A}_k^{\dagger 2}\bar{A}_k^2\sim O(1/L), \quad (67)$$

where we have used $|\bar{c}_{k1}|^4\sim|V_k|^4\sim O(1/L^2)$ and the fact that $\langle\bar{A}_k^{\dagger 2}\bar{A}_k^2\rangle\sim O(L^0)$ in the thermodynamic limit. This last relation follows from Eq. (14), together with

$$\langle\dot{\tilde{A}}_k^\dagger\tilde{A}_k\rangle=\langle a_k^\dagger a_k\rangle+O(1/\sqrt{L}), \quad (68)$$

which is due to the volume dependence of the interaction $V_k$. With this consideration, after suitable relabeling $\tilde{l}=l-1$ and $\tilde{l}'=l'-1$, Eq. (66) becomes

$$\bar{\Pi}^\dagger\left[(m-1)\left(\sum_k\bar{D}_k^{\dagger 2}\right)\sum_{\tilde{l}=0}^{m-2}C_{\tilde{l}}^{m-2}\bar{B}_1^{\dagger m-\tilde{l}-2}\left(\sum_{k'}\bar{D}_{k'}^\dagger\right)^{\tilde{l}}\left(\bar{B}_1+\sum_k\bar{D}_k\right)^n\right] + \bar{\Pi}^\dagger\left[n\left(\sum_k\bar{D}_k^\dagger\bar{D}_k\right)\left(\bar{B}_1^\dagger+\sum_k\bar{D}_k^\dagger\right)^{m-1}\sum_{\tilde{l}'=0}^{n-1}C_{\tilde{l}}^{n-1}\bar{B}_1^{n-1-\tilde{l}}\left(\sum_{k'}\bar{D}_{k'}^\dagger\right)^{\tilde{l}'}\right]$$
$$=(m-1)\bar{X}\bar{\Pi}^\dagger\left[\left(\bar{B}_1^\dagger+\sum_k\bar{D}_k^\dagger\right)^{m-2}\left(\bar{B}_1+\sum_k\bar{D}_k\right)^n\right]+n\bar{Y}\bar{\Pi}^\dagger\left[\left(\bar{B}_1^\dagger+\sum_k\bar{D}_k^\dagger\right)^{m-1}\left(\bar{B}_1+\sum_k\bar{D}_k\right)^{n-1}\right]$$
$$=(m-1)\bar{X}\bar{\Pi}^\dagger(a_1^{\dagger m-2}a_1^n)+n\bar{Y}\bar{\Pi}^\dagger(a_1^{\dagger m-1}a_1^{n-1}) \quad (69)$$

where

$$\bar{X}=\bar{\Pi}^\dagger\left(\sum_k\bar{D}_k^{\dagger 2}\right) \quad (70)$$

and

$$\bar{Y}=\bar{\Pi}^\dagger\left(\sum_k\bar{D}_k^\dagger\bar{D}_k\right), \quad (71)$$

which give Eqs. (59) and (60), respectively.

With Eqs. (65) and (69), we have proven Eq. (57). Relation (58) can be proved in a similar way.

### VI. Π IN THE NONINTEGRABLE CASE

In the nonintegrable case (b) discussed in Sec. III, the particle frequency $\omega_1$ can resonate with the frequencies $\omega_k$ of the field. There is no self-adjoint perturbed invariant $A_1^\dagger A_1$ corresponding to the unperturbed invariant $a_1^\dagger a_1$, which is expandable around $\lambda=0$. This could be expected, because now the particle is a damped oscillator. Damping comes from resonant emission of field modes. Due to damping there is no invariant of the form $A_1^\dagger A_1$ [22]. The Hamiltonian can now be written as [12,14]

$$H=\sum_{k=0}^{\infty}\omega_k\tilde{A}_k^\dagger\tilde{A}_k+\tilde{E}_{\rm vac}, \quad (72)$$

where $\tilde{A}_k$ and $\tilde{A}_k^\dagger$ are renormalized annihilation and creation operators of the field and $\tilde{E}_{\rm vac}$ is the renormalized vacuum energy in the nonintegrable case. As in scattering theory we can choose either "in" or "out" operators [15]. Hereafter we will use out operators. As we will see, from the out operators we will obtain damping for $t>0$ in the Heisenberg picture. The explicit form of the operator $\widetilde{A}_k$ is

$$\widetilde{A}_k = \widetilde{c}_{k1}a_1 + \widetilde{d}_{k1}a_1^\dagger + \sum_{k'} \widetilde{c}_{kk'}a_{k'} + \sum_{k'} \widetilde{d}_{kk'}a_{k'}^\dagger, \tag{73}$$

with the coefficients given in Appendix A. These coefficients are proportional to the Green's function $G^+(\omega_k)$ where

$$G^\pm(z) = \left(\omega_1^2 - z^2 - \int_0^\infty dk \frac{4\omega_1\omega_k\lambda^2 v_k^2}{(\omega_k^2 - z^2)_\pm}\right)^{-1}, \tag{74}$$

for general complex argument $z$ with $\mathrm{Re}(z)>\omega_0$. The + (−) sign means the function is analytically continued from the upper (lower) sheet of $z$. The function $G^+(z)$ has a pole on the "second sheet," obtained by analytic continuation from the upper to the lower half plane of $z$ across the branch cut on the positive real axis. Denoting this pole as

$$z_1 \equiv \widetilde{\omega}_1 - i\gamma \tag{75}$$

(with $\gamma>0$) we have $G^+(z_1)^{-1}=0$. This pole reduces to $\omega_1$ when $\lambda\to 0$.

By extracting the residue at this pole in Eq. (72) we obtain the complex spectral representation (see Ref. [15])

$$H = z_1 A_1^\dagger \widetilde{A}_1 + \sum_{k=0}^{\infty} \omega_k \widetilde{A}_k^\dagger A_k + \widetilde{E}_{\mathrm{vac}}, \tag{76}$$

where

$$A_k = \widetilde{A}_k[1 + 2\pi i(\omega_k - z_1)\delta_c(\omega_k - z_1)], \tag{77}$$

$\delta_c$ is the complex $\delta$ function, and

$$\begin{aligned} z_1 A_1^\dagger \widetilde{A}_1 &= -\mathrm{Res}(\omega_k \widetilde{A}_k^\dagger \widetilde{A}_k)_{\omega_k=z_1} \\ &= -\sum_{k=0}^{\infty} \omega_k \widetilde{A}_k^\dagger \widetilde{A}_k 2\pi i(\omega_k - z_1)\delta_c(\omega_k - z_1). \end{aligned} \tag{78}$$

To evaluate the complex $\delta$ function we first go to the continuous limit so the summations go to integrals. Then we deform the integration path to a small contour surrounding the pole $z_1$.

By separating the residue at the pole $z_1$ (or $z_1^*$) we obtain the particle operators $A_1^\dagger$, $\widetilde{A}_1$ in the complex spectral representation of the Hamiltonian. In this way we obtain a closer correspondence between the integrable and nonintegrable cases. Note that the complex $\delta$ function is a generalized function.

In terms of the nonunitary transformation $\Lambda$ mentioned in Sec. III we have

$$\begin{aligned} A_i^\dagger &= \Lambda^{-1} a_i^\dagger, \quad \widetilde{A}_i^\dagger = \Lambda^\dagger a_i^\dagger, \\ A_i &= \Lambda^{-1} a_i, \quad \widetilde{A}_i = \Lambda^\dagger a_i. \end{aligned} \tag{79}$$

The explicit forms of the new operators in Eq. (76) are

$$A_1^\dagger = c_{11}^* a_1^\dagger + d_{11}^* a_1 + \sum_k c_{1k}^* a_k^\dagger + \sum_k d_{1k}^* a_k,$$

$$\widetilde{A}_1 = c_{11}^* a_1 + d_{11}^* a_1^\dagger + \sum_k c_{1k}^* a_k + \sum_k d_{1k}^* a_k^\dagger,$$

$$A_k = c_{k1}a_1 + d_{k1}a_1^\dagger + \sum_{k'} c_{kk'}a_{k'} + \sum_{k'} d_{kk'}a_{k'}^\dagger, \tag{80}$$

with the coefficients presented in Appendix A. The transformed operators satisfy the relations

$$\begin{aligned} L_H \widetilde{A}_1^\dagger &= z_1^* \widetilde{A}_1^\dagger, \quad L_H \widetilde{A}_1 = -z_1 \widetilde{A}_1, \\ L_H A_1^\dagger &= z_1 A_1^\dagger, \quad L_H A_1 = -z_1^* A_1, \\ L_H \widetilde{A}_k^\dagger &= \omega_k \widetilde{A}_k^\dagger, \quad L_H \widetilde{A}_k = -\omega_k \widetilde{A}_k. \end{aligned} \tag{81}$$

Due to the complex $\delta$ function, the operators in Eq. (80) do not preserve the Hilbert space. For example one can show that (see Appendix B)

$$[\widetilde{A}_1, \widetilde{A}_1^\dagger] = 0 \tag{82}$$

and similarly $[A_1, A_1^\dagger]=0$.

Provided the test functions for integration do not contain singularities at $\omega_k=z_1$ or $\omega_k=z_1^*$, the new set of operators obey the commutation relations [15]

$$\begin{aligned} &[\widetilde{A}_1, A_1^\dagger] = 1, \\ &[\widetilde{A}_k, \widetilde{A}_{k'}^\dagger] = [\widetilde{A}_k, A_{k'}^\dagger] = [A_k, A_{k'}^\dagger] = \delta_{k,k'}. \end{aligned} \tag{83}$$

Other commutators are zero. If the test functions contain singularities, then we need a careful consideration [12]. Two examples are presented in Appendix B. Hereafter we assume that the density operator $\rho$ gives no such singularities.

From Eq. (73) we have

$$a_1^\dagger = \sum_k \widetilde{D}_k^\dagger \tag{84}$$

where

$$\widetilde{D}_k^\dagger = \widetilde{c}_{k1}\widetilde{A}_k^\dagger - \widetilde{d}_{k1}^* \widetilde{A}_k. \tag{85}$$

Separating the poles at $\omega_k=z_1$, $z_1^*$ from Eq. (84) we get

$$a_1^\dagger = \widetilde{B}_1^\dagger + \sum_k D_k^\dagger \tag{86}$$

where

$$\widetilde{B}_1^\dagger = c_{11}\widetilde{A}_1^\dagger - d_{11}^* \widetilde{A}_1, \tag{87}$$

$$D_k^\dagger = \widetilde{c}_{k1} A_k^\dagger - \widetilde{d}_{k1}^* A_k. \tag{88}$$

Note that from Eq. (84) we can calculate the exact time evolution of $a_1^\dagger$ as

$$e^{iL_H t} a_1^\dagger = \sum_k (\widetilde{c}_{k1} e^{i\omega_k t} \widetilde{A}_k^\dagger - \widetilde{d}_{k1}^* e^{-i\omega_k t} \widetilde{A}_k). \tag{89}$$

From Eq. (89) we can calculate the exact time evolution of any observable associated with the particle, for example its energy. Our goal though is to extract the "kinetic" part of the time evolution, which follows a closed, exact Markovian dynamics. This is why we introduce the projector $\Pi$ (or $\Pi^\dagger$).

We will calculate the explicit form of the $\Pi^\dagger$ projector acting on products of creation and annihilation operators by extending the results of the integrable case. As in the integrable case, the projection $\Pi^\dagger(a_1^{\dagger m}a_1^n)$ should keep terms where the creation operators $\widetilde{A}_k^\dagger$ are paired with the destruction operators $\widetilde{A}_k$. At the same time, $\Pi^\dagger$ should leave intact particle operators $\widetilde{A}_1^\dagger$ and $\widetilde{A}_1$. As we have seen, the latter are residues at the poles $\omega_k=z_1^*,z_1$.

To define $\Pi^\dagger$ we start by writing [see Eq. (84)]

$$a_1^{\dagger m}a_1^n=\sum_{k_1\cdots k_{m+n}}\widetilde{D}_{k_1}^\dagger\cdots\widetilde{D}_{k_m}^\dagger\widetilde{D}_{k_{m+1}}\cdots\widetilde{D}_{k_{m+n}}. \tag{90}$$

We decompose this into a sum of all possible pairings. For example for $m=n=2$ we have

$$\begin{aligned}a_1^{\dagger 2}a_1^2=&\sum_{k_1\cdots k_4}{}'\widetilde{D}_{k_1}^\dagger\widetilde{D}_{k_2}^\dagger\widetilde{D}_{k_3}\widetilde{D}_{k_4}+\sum_{k_1,k_3,k_4}{}'\widetilde{D}_{k_1}^\dagger\widetilde{D}_{k_1}^\dagger\widetilde{D}_{k_3}\widetilde{D}_{k_4}\\&+\sum_{k_1,k_2,k_4}{}'\widetilde{D}_{k_1}^\dagger\widetilde{D}_{k_2}^\dagger\widetilde{D}_{k_2}\widetilde{D}_{k_4}+\cdots+\sum_{k_1,k_2}{}'\widetilde{D}_{k_1}^\dagger\widetilde{D}_{k_2}^\dagger\widetilde{D}_{k_1}\widetilde{D}_{k_2}\\&+\cdots,\end{aligned} \tag{91}$$

where the prime in the summations means that no summation variables are equal. Once we have done this separation we can extract the poles of the unmatched operators using Eq. (86). For example we have

$$\begin{aligned}\sum_{k_1,k_2,k_4}{}'\widetilde{D}_{k_1}^\dagger\widetilde{D}_{k_2}^\dagger\widetilde{D}_{k_2}\widetilde{D}_{k_4}=&\sum_{k_2}{}'\left(\widetilde{B}_1^\dagger+\sum_{k_1}D_{k_1}^\dagger\right)\\&\times\widetilde{D}_{k_2}^\dagger\widetilde{D}_{k_2}\left(\widetilde{B}_1+\sum_{k_4}D_{k_4}\right).\end{aligned} \tag{92}$$

To get the $\Pi^\dagger$ projection, we simply drop the unmatched field operators. Thus we have

$$\Pi^\dagger\sum_{k_1,k_2,k_4}{}'\widetilde{D}_{k_1}^\dagger\widetilde{D}_{k_2}^\dagger\widetilde{D}_{k_2}\widetilde{D}_{k_4}=\sum_{k_2}\widetilde{B}_1^\dagger\widetilde{D}_{k_2}^\dagger\widetilde{D}_{k_2}\widetilde{B}_1. \tag{93}$$

Since the operators with different $k_i$ in the left hand side of Eq. (92) commute, the operators with different index in the right hand side of Eq. (93) also commute.

In general we can write

$$\Pi^\dagger(a_1^{\dagger m}a_1^n)=\Pi^\dagger\left[\left(\widetilde{B}_1^\dagger+\sum_k\widetilde{D}_k^\dagger\right)^m\left(\widetilde{B}_1+\sum_k\widetilde{D}_k\right)^n\right] \tag{94}$$

where the projection in Eq. (94) is defined as follows:

$$\Pi^\dagger\left(\widetilde{A}_1^{\dagger m_1}\widetilde{A}_1^{n_1}\prod_{k=0}^{\infty}\widetilde{A}_k^{\dagger m_k}\widetilde{A}_k^{n_k}\right)=\widetilde{A}_1^{\dagger m_1}\widetilde{A}_1^{n_1}\prod_k\delta_{m_k,n_k}\widetilde{A}_k^{\dagger m_k}\widetilde{A}_k^{m_k}, \tag{95}$$

which corresponds to Eq. (56) in the integrable case.

In the Heisenberg picture, Eq. (95) decays for $t>0$ and $m_1$, $n_1>0$. If we had started with the in operators we would obtain decay for $t<0$. For $m_1=n_1=0$, Eq. (95) remains invariant.

From Eq. (72) we see that $\Pi^\dagger H=H$; hence

$$\mathrm{Tr}(H\rho)=\mathrm{Tr}(H\Pi\rho), \tag{96}$$

which shows that $\rho_{\mathrm{eq}}=\Pi\rho_{\mathrm{eq}}$ for the equilibrium distribution. Similarly to Eq. (96), for any invariant observable $I$ we have

$$\mathrm{Tr}(I\rho)=\mathrm{Tr}(I\Pi\rho). \tag{97}$$

Therefore the state $\Pi\rho$ contains the total energy and probability of $\rho$. For noninvariant observables, $\Pi\rho$ extracts the purely exponential terms of the time evolution, which are associated with the complex energies $z_1$, $z_1^*$. In this way, $\Pi\rho$ gives the Markovian dynamics of approach to equilibrium. The complementary component $\hat{\Pi}\rho$ extracts nonexponential terms that give memory effects. This non-Markovian component contains no net energy or probability.

Following the same steps as in the integrable case we obtain recursive relations corresponding to Eqs. (57) and (58),

$$\begin{aligned}\Pi^\dagger(a_1^{\dagger m+1}a_1^n)=&\Pi^\dagger a_1^\dagger\Pi^\dagger(a_1^{\dagger m}a_1^n)+mX\Pi^\dagger(a_1^{\dagger m-1}a_1^n)\\&+nY\Pi^\dagger(a_1^{\dagger m}a_1^{n-1}),\end{aligned} \tag{98}$$

$$\begin{aligned}\Pi^\dagger(a_1^{\dagger m}a_1^{n+1})=&\Pi^\dagger(a_1^{\dagger m}a_1^n)\Pi^\dagger a_1+nX\Pi^\dagger(a^{\dagger m}a_1^{n-1})\\&+mY\Pi^\dagger(a_1^{\dagger m-1}a_1^n)\end{aligned} \tag{99}$$

where

$$X=-\sum_k\widetilde{c}_{k1}\widetilde{d}_{k1}^*\{\widetilde{A}_k,\widetilde{A}_k^\dagger\}, \tag{100}$$

$$Y=\sum_k|\widetilde{c}_{k1}|^2\widetilde{A}_k^\dagger\widetilde{A}_k+|\widetilde{d}_{k1}|^2\widetilde{A}_k\widetilde{A}_k^\dagger. \tag{101}$$

The $\Pi$ superoperator we introduced satisfies all the conditions of Sec. II. The validity of condition (A) is self-evident from the identification of $\Pi^\dagger$ with the projector in (95).

The second condition (B) can be verified from the relations

$$L_H\mathcal{A}=z\mathcal{A},\quad \Pi^\dagger\mathcal{A}=\xi\mathcal{A}, \tag{102}$$

with

$$\mathcal{A}=\widetilde{A}_1^{\dagger m_1}\widetilde{A}_1^{n_1}\prod_k\widetilde{A}_k^{\dagger m_k}\widetilde{A}_k^{n_k},$$

$$z=m_1z_1^*-n_1z_1+\sum_k(m_k-n_k)\omega_k,$$

$$\xi=0,1. \tag{103}$$

This implies that $\mathrm{Tr}(\mathcal{A}[L_H,\Pi]\rho)=0$ for all observables $\mathcal{A}$ and their linear combinations.

The third condition (C) is verified in Appendix C using the recursive relations for $\Pi^\dagger$.

## VII. EXACT MARKOVIAN KINETIC EQUATION

With all the previous preparations, we are ready now to derive the explicit form of the Markovian equation (32). As we saw in Sec. II, in order to obtain this equation we need to evaluate $L_H\Pi^\dagger W^\dagger = L_H\Pi^\dagger(a_1^{\dagger m}a_1^n)$.

From the recursive relation (98) we get

$$L_H\Pi^\dagger(a_1^{\dagger m}a_1^n) = L_H\tilde{B}_1^\dagger \cdot \Pi^\dagger(a_1^{\dagger m-1}a_1^n) + \tilde{B}_1^\dagger \cdot L_H\Pi^\dagger(a_1^{\dagger m-1}a_1^n) + (m-1)XL_H\Pi^\dagger(a_1^{\dagger m-2}a_1^n) + nYL_H\Pi^\dagger(a_1^{\dagger m-1}a_1^{n-1}), \quad (104)$$

where $L_HX=0=L_HY$ since $X$ and $Y$ are diagonal in the transformed creation-annihilation operators of the field.

Writing

$$\tilde{A}_1^\dagger = \frac{1}{\Delta}(c_{11}^*\tilde{B}_1^\dagger + d_{11}^*\tilde{B}_1), \quad \tilde{A}_1 = \frac{1}{\Delta}(c_{11}\tilde{B}_1 + d_{11}\tilde{B}_1^\dagger), \quad (105)$$

where

$$\Delta = |c_{11}|^2 - |d_{11}|^2 = |N|^2\frac{\tilde{\omega}_1}{\omega_1}, \quad (106)$$

and using Eq. (81), we find that

$$L_H\tilde{B}_1^\dagger = \alpha^*\tilde{B}_1^\dagger + \beta\tilde{B}_1,$$

$$L_H\tilde{B}_1 = -\beta^*\tilde{B}_1^\dagger - \alpha\tilde{B}_1, \quad (107)$$

where

$$\alpha = \frac{\tilde{\omega}_1^2 + \gamma^2 + \omega_1^2}{2\omega_1} - i\gamma,$$

$$\beta = \frac{\tilde{\omega}_1^2 + \gamma^2 - \omega_1^2}{2\omega_1} - i\gamma. \quad (108)$$

From Eq. (104) we then infer that

$$L_H\Pi^\dagger(a_1^{\dagger m}a_1^n) = (m\alpha^* - n\alpha)\Pi^\dagger(a_1^{\dagger m}a_1^n) + m\beta\Pi^\dagger(a_1^{\dagger m-1}a_1^{n+1}) - n\beta^*\Pi^\dagger(a_1^{\dagger m+1}a_1^{n-1}) - mn[(\alpha^*-\alpha)Y + (\beta-\beta^*)X]\Pi^\dagger(a_1^{\dagger m-1}a_1^{n-1}) - m(m-1)[\alpha^*X + \beta Y]\Pi^\dagger(a_1^{\dagger m-2}a_1^n) + n(n-1)[\alpha X + \beta^*Y]\Pi^\dagger(a_1^{\dagger m}a_1^{n-2}). \quad (109)$$

The consistency between Eqs. (104) and (109) can be proven by first inserting the recursive relations for $\Pi^\dagger(a_1^{\dagger m}a_1^n)$, i.e., Eqs. (98) and (99), together with Eq. (109) into the right hand side of Eq. (104). Then we verify that the coefficients of $\Pi^\dagger(a_1^{\dagger m}a_1^n)$ for all values of $m$ and $n$ on the right hand side of Eq. (104) are identical to the corresponding coefficients in Eq. (109).

Now we come to our kinetic equation. We have the relations

$$\Pi^\dagger(a_1^{\dagger m}a_1^n \cdot Y) = Y \cdot \Pi^\dagger(a_1^{\dagger m}a_1^n),$$

$$\Pi^\dagger(a_1^{\dagger m}a_1^n \cdot X) = X \cdot \Pi^\dagger(a_1^{\dagger m}a_1^n), \quad (110)$$

because $Y$ and $X$ are diagonal in the field operators, and $\Pi^\dagger$ is a projection of the diagonal component of the field operators.

Using Eqs. (110), we write Eq. (109) as

$$L_H\Pi^\dagger(a_1^{\dagger m}a_1^n) = \Pi^\dagger\Xi \quad (111)$$

where

$$\Xi = (m\alpha^* - n\alpha)(a_1^{\dagger m}a_1^n) + m\beta(a_1^{\dagger m-1}a_1^{n+1}) - n\beta^*(a_1^{\dagger m+1}a_1^{n-1}) - mn[(\alpha^*-\alpha)Y + (\beta-\beta^*)X](a_1^{\dagger m-1}a_1^{n-1}) - m(m-1)(\alpha^*X + \beta Y)(a_1^{\dagger m-2}a_1^n) + n(n-1)(\alpha X + \beta^*Y)(a_1^{\dagger m}a_1^{n-2}).$$

Defining

$$\tilde{\rho} = \Pi\rho, \quad (112)$$

the kinetic equation (32) becomes

$$i\frac{\partial}{\partial t}\mathrm{Tr}(W\tilde{\rho}) = \mathrm{Tr}(\Xi^\dagger\tilde{\rho}). \quad (113)$$

Using the identities for the trace in Appendix F, and defining $\alpha_R = \mathrm{Re}(\alpha)$, $\beta_R = \mathrm{Re}(\beta)$, we obtain

$$i\frac{\partial}{\partial t}\mathrm{Tr}(W\tilde{\rho}) = \mathrm{Tr}\Bigg(W \cdot \Bigg\{\alpha_R([a_1^\dagger a_1,\tilde{\rho}]) - i\gamma\left(Y - X + \frac{1}{2}\right) \times([a_1\tilde{\rho},a_1^\dagger] + [a_1,\tilde{\rho}a_1^\dagger] + [a_1^\dagger\tilde{\rho},a_1] + [a_1^\dagger,\tilde{\rho}a_1]) + i\frac{\gamma}{2}([a_1\tilde{\rho},a_1^\dagger] + [a_1,\tilde{\rho}a_1^\dagger] - [a_1^\dagger\tilde{\rho},a_1] - [a_1^\dagger,\tilde{\rho}a_1]) + \beta_R([a_1^\dagger,\tilde{\rho}a_1^\dagger] - [a_1\tilde{\rho},a_1]) + i\gamma([a_1^\dagger,\tilde{\rho}a_1^\dagger] + [a_1\tilde{\rho},a_1]) + (\alpha_RX + \beta_RY)([a_1^\dagger\tilde{\rho},a_1^\dagger] + [a_1^\dagger,\tilde{\rho}a_1^\dagger] - [a_1\tilde{\rho},a_1] - [a_1,\tilde{\rho}a_1]) + i\gamma(Y-X)([a_1^\dagger\tilde{\rho},a_1^\dagger] + [a_1^\dagger,\tilde{\rho}a_1^\dagger] + [a_1\tilde{\rho},a_1] + [a_1,\tilde{\rho}a_1])\Bigg\}\Bigg). \quad (114)$$

In this form we can already see an interesting property of this equation: it is closed for the component $P\tilde{\rho}$, as mentioned after Eq. (34). To see this, note that $\mathrm{Tr}(W\tilde{\rho}) = \mathrm{Tr}(WP\tilde{\rho})$. Moreover we have $[P,Y] \sim [P,X] \sim O(1/\sqrt{L})$. Thus in the thermodynamic limit we can move $P$ past $X$ and $Y$. By its definition, we can also move $P$ past $a_1$, $a_1^\dagger$. So in Eq. (114) we can replace $\tilde{\rho}$ by $P\tilde{\rho}$. This shows that this is a closed equation for $P\tilde{\rho}$.

The equation takes a simpler form if we assume that initially the density matrix is factored into particle and field components,

$$P\tilde{\rho}(0) = \tilde{\rho}_1(0) \times \prod_k \tilde{\rho}_k(0). \quad (115)$$

To see this, we write Eq. (114) as

$$i\frac{\partial}{\partial t}\mathrm{Tr}(W\tilde{\rho}) = \mathrm{Tr}(W\theta\tilde{\rho}) \quad (116)$$

with the formal solution

$$\mathrm{Tr}[W\tilde{\rho}(t)] = \mathrm{Tr}[We^{-i\theta t}\tilde{\rho}(0)]. \quad (117)$$

The collision superoperator has the following operator dependence:

$$\theta = \theta(X, Y, a_1^\dagger, a_1). \tag{118}$$

Following an argument similar to the one above Eq. (67) (see also Ref. [23]), we may neglect correlations for $Y$ and $X$ products, i.e.,

$$\langle Y^n X^m \rangle = \langle Y \rangle^n \langle X \rangle^m + O(1/L), \tag{119}$$

where using Eqs. (68) and (115) we have

$$\langle Y \rangle = \mathrm{Tr}_{\mathrm{F}}\left( Y \prod_k \tilde{\rho}_k(0) \right), \tag{120}$$

and similarly for $X$. Thus in Eq. (117) we can replace $X$ and $Y$ by their initial averages:

$$\theta = \theta(\langle X \rangle, \langle Y \rangle, a_1^\dagger, a_1). \tag{121}$$

This means that, neglecting $O(1/L)$ terms, only the particle component of the density operator evolves in time,

$$P\tilde{\rho}(t) = \tilde{\rho}_1(t) \times \prod_k \tilde{\rho}_k(0), \tag{122}$$

and the field density operator drops out after taking the trace. Using this result and defining the reduced density matrix

$$\tilde{\rho}_1(x_1, x_1') = \langle x_1 | \tilde{\rho}_1 | x_1' \rangle \tag{123}$$

we write the kinetic equation in the dimensionless coordinate representation as (see Appendix G)

$$\begin{aligned} i\frac{\partial}{\partial t}\tilde{\rho}_1 = \Bigg\{ &-\frac{\alpha_R}{2}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x'^2_1}\right) + \frac{\alpha_R}{2}(x_1^2 - x'^2_1) + i\gamma\left(\langle Y \rangle - \langle X \rangle + \frac{1}{2}\right)\left[\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x'_1}\right)^2 - (x_1 - x'_1)^2\right] + i\frac{\gamma}{2}\left[\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x'_1}\right)\right. \\ &\left.\times (x_1 + x'_1) - (x_1 - x'_1)\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x'_1}\right)\right] + \frac{\beta_R}{2}\left[(x_1^2 - x'^2_1) + \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x'^2_1}\right) + 2(x_1 - x'_1) + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x'_1}\right)\right] \\ &- i\frac{\gamma}{2}\left[(x_1 - x'_1)^2 + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x'_1}\right)^2 + (x_1 - x'_1)\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x'_1}\right) + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x'_1}\right)(x_1 + x'_1)\right] \\ &+ (\alpha_R \langle X \rangle + \beta_R \langle Y \rangle)\left[2(x_1 - x'_1)\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x'_1}\right)\right] + i\gamma(\langle X \rangle - \langle Y \rangle)\left[(x_1 - x'_1)^2 + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x'_1}\right)^2\right]\Bigg\}\tilde{\rho}_1, \end{aligned} \tag{124}$$

since it is true when averaging over arbitrary polynomial $W$. Recollecting terms we finally have our exact Markovian equation

$$\begin{aligned} i\frac{\partial}{\partial t}\tilde{\rho}_1 = &\frac{|z_1|^2}{2\omega_1}(x_1^2 - x'^2_1)\tilde{\rho} - \frac{\omega_1}{2}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x'^2_1}\right)\tilde{\rho}_1 \\ &- i\gamma(x_1 - x'_1)\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x'_1}\right)\tilde{\rho}_1 - 2i\gamma K(x_1 - x'_1)^2\tilde{\rho}_1 \\ &+ \omega_1 J(x_1 - x'_1)\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x'_1}\right)\tilde{\rho}_1. \end{aligned} \tag{125}$$

The coefficients are given by

$$K = \langle Y \rangle - \langle X \rangle + \frac{1}{2},$$

$$J = \frac{|z_1|^2}{\omega_1^2}J' - K,$$

$$J' = \langle Y \rangle + \langle X \rangle + \frac{1}{2}. \tag{126}$$

Using the identities [14]

$$\sum_k \omega_k(|\tilde{c}_{k1}|^2 + |\tilde{d}_{k1}|^2) = \omega_1, \quad \sum_k \omega_k \tilde{c}^*_{k1}\tilde{d}_{k1} = 0, \tag{127}$$

we get

$$K = 4\int_0^\infty dk\, \lambda^2 v_k^2 |G^+(\omega_k)|^2 \omega_k^2 \left(n_k + \frac{1}{2}\right),$$

$$J' = 4\int_0^\infty dk\, \lambda^2 v_k^2 |G^+(\omega_k)|^2 \omega_1^2 \left(n_k + \frac{1}{2}\right),$$

$$J = 4\int_0^\infty dk\, \lambda^2 v_k^2 |G^+(\omega_k)|^2 (|z_1|^2 - \omega_k^2)\left(n_k + \frac{1}{2}\right), \tag{128}$$

where $n_k = \langle a_k^\dagger a_k \rangle$.

Note that for the integrable case, with no particle-field resonance, we have $\gamma = 0$ and the damping and diffusion terms in Eq. (125) vanish. Still, the last term remains. This corresponds to the excitation of the particle due to virtual processes.

We also note that the kinetic equation (114) is valid for all field distributions, while Eq. (125) is valid for field distributions of the form (115), which are more general than Gibbsian distributions.

## VIII. COMPARISON WITH THE HU-PAZ-ZHANG KINETIC EQUATION

In this section we will obtain an expression for the kinetic equation correct up to $O(\lambda^2)$, in the weak coupling limit $\lambda \to 0$. We then compare this to the $\lambda^2 t$ limit of the HPZ equation and show that they are identical. The $\lambda^2 t$ limit means that we take $\lambda \to 0$, $t\to\infty$ with $\lambda^2 t$ finite. Physically, this means that we consider times of the order of the relaxation time $t_{\rm rel} \sim 1/\lambda^2$ in a weakly coupled system.

We start by assuming the harmonic oscillator is interacting with a bath of field modes that is in thermal equilibrium, and the number density of the field degrees of freedom $n_k$ satisfies the Planck distribution. We then have

$$n_k + \frac{1}{2} = \frac{1}{e^{\beta\hbar\omega_k}-1} + \frac{1}{2} = \frac{1}{2}\coth\left(\frac{\beta\hbar\omega_k}{2}\right), \qquad (129)$$

where $\beta = 1/k_B T$ and $k_B$ is the Boltzmann constant.

In what follows, we are going to use the approximation of the function $G^{\pm}(\omega_k)$ shown in Appendix E. First we consider the coefficient $J$,

$$J = -4\int_0^\infty dk\, \lambda^2 v_k^2 |G_k^+|^2 (w_k^2 - |z_1|^2)\left(n_k + \frac{1}{2}\right). \qquad (130)$$

From Eq. (E9) we find that in the zeroth order approximation

$$\lambda^2 v_k^2 |G^+(\omega_k)|^2 (\omega_k^2 - \omega_1^2) = 0. \qquad (131)$$

The next order can be found using the expression with $\epsilon$ an infinitesimal [see Eq. (E5)]

$$|G^+(\omega_k)|^2 \simeq \frac{1}{(\omega_k^2-\omega_1^2)^2 + \epsilon^2}, \qquad (132)$$

which gives

$$\lambda^2 v_k^2 |G^+(\omega_k)|^2 (\omega_k^2 - \omega_1^2) \simeq \lambda^2 v_k^2 \frac{\omega_k^2-\omega_1^2}{(\omega_k^2-\omega_1^2)^2+\epsilon^2} = \lambda^2 v_k^2 \mathcal{P}\frac{1}{\omega_k^2-\omega_1^2}. \qquad (133)$$

Thus we have

$$J \simeq -2\int_0^\infty dk\, \lambda^2 v_k^2 \mathcal{P}\left(\frac{1}{\omega_k^2-\omega_1^2}\right)\coth\left(\frac{\beta\hbar\omega_k}{2}\right). \qquad (134)$$

For the coefficient $K$ we have using Eq. (E9)

$$K \simeq \frac{1}{2\omega_1^2}\int_0^\infty d\omega_k \delta(\omega_k-\omega_1)\omega_k^2 \coth\left(\frac{\beta\hbar\omega_k}{2}\right) = \frac{1}{2}\coth\left(\frac{\beta\hbar\omega_1}{2}\right). \qquad (135)$$

Therefore, in the $\lambda^2$ approximation, the kinetic equation is

$$i\frac{\partial}{\partial t}\tilde{\rho} = \Bigg[-\frac{\omega_1}{2}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial {x'}_1^2}\right) + \frac{\tilde{\omega}_1^2}{2\omega_1}(x_1^2 - {x'}_1^2) - i\gamma(x_1 - x_1') \times\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial {x'}_1}\right) - i\gamma\coth\left(\frac{\beta\hbar\omega_k}{2}\right)(x_1-x_1')^2 - \mathcal{P}\int_0^\infty dk \frac{2\omega_1\lambda^2 v_k^2}{\omega_k^2-\omega_1^2}\coth\left(\frac{\beta\hbar\omega_1}{2}\right) \times (x_1 - x_1')\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial {x'}_1}\right)\Bigg]\tilde{\rho}. \qquad (136)$$

Now we show that this expression is exactly the same as the $\lambda^2 t$ limit of the HPZ equation. For weak coupling, the time-dependent coefficients of the HPZ equation (2) are found to be [2]

$$\tilde{\Omega}^2(t) \simeq \omega_1^2 + \delta\Omega^2(t), \qquad (137)$$

$$\delta\Omega^2(t) \simeq 2\int_0^t ds\ \eta(s)\cos(\omega_1 s), \qquad (138)$$

$$\Gamma(t) \simeq -\frac{1}{\omega_1}\int_0^t ds\ \eta(s)\sin(\omega_1 s), \qquad (139)$$

$$\Gamma(t)h(t) \simeq \int_0^t ds\ \nu(s)\cos(\omega_1 s), \qquad (140)$$

$$\Gamma(t)f(t) \simeq \frac{1}{\omega_1}\int_0^t ds\ \nu(s)\sin(\omega_1 s), \qquad (141)$$

where

$$\eta(s) = -\int_0^\infty d\omega\, I(\omega)\sin(\omega_1 s), \qquad (142)$$

$$\nu(s) = \int_0^\infty d\omega\, I(\omega)\coth\left(\frac{\beta\hbar\omega}{2}\right)\cos(\omega_1 s), \qquad (143)$$

$$I(\omega) = \frac{1}{M_1}\sum_{k=0}^{\infty}\delta(\omega-\omega_k)\frac{\lambda^2 C_k^2}{2M_k\omega_k} = 2\omega_1\sum_{k=0}^{\infty}\delta(\omega-\omega_k)\lambda^2 V_k^2 = 2\omega_1\lambda^2\bar{v}^2(\omega). \qquad (144)$$

We note that we have added in an extra factor of $1/M_1$ in the expression of $I(\omega)$ compared to Ref. [2] due to the different definition of the position $x_1$.

In the $\lambda^2 t$ limit we then take $t\to\infty$. We use the trick in [11] to evaluate the time integrations. For example we have

$$\int_0^\infty ds\,\sin(\omega_1 s)\sin(\omega s) = \lim_{\epsilon\to 0^+}\frac{-1}{4}\int_0^\infty ds\left(e^{i(\omega_1+\omega+i\epsilon)s} + e^{-i(\omega_1+\omega-i\epsilon)s} - e^{i(\omega_1-\omega+i\epsilon)s} - e^{-i(\omega_1-\omega-i\epsilon)s}\right)$$
$$= -\frac{1}{4}\left(\frac{-1}{i(\omega_1+\omega+i\epsilon)} - \frac{1}{-i(\omega_1+\omega-i\epsilon)} + \frac{1}{i(\omega_1-\omega+i\epsilon)} + \frac{1}{-i(\omega_1-\omega-i\epsilon)}\right)$$
$$= -\frac{1}{4i}[2i\pi\delta(\omega_1+\omega) - 2i\pi\delta(\omega_1-\omega)] = \frac{\pi}{2}\delta(\omega_1-\omega). \quad (145)$$

The last line is due to the fact that $\omega$ can only be positive.

In this way we can calculate all the coefficients. Using the expressions for $\tilde{\omega}_1$ and $\gamma$ in Appendix D we find

$$\delta\Omega^2(t) \simeq -2\int_0^\infty d\omega\, I(\omega)\mathcal{P}\frac{\omega}{\omega^2-\omega_1^2} = 4\int_0^\infty dk\,\mathcal{P}\frac{\omega_1\omega_k\lambda^2 v_k^2}{\omega_1^2-\omega_k^2}$$
$$= \tilde{\omega}_1^2 - \omega_1^2, \quad (146)$$

$$\Gamma(t) \simeq \frac{1}{\omega_1}\int_0^\infty d\omega\, I(\omega)\frac{\pi}{2}\delta(\omega_1-\omega) = \frac{\pi}{2\omega_1}I(\omega_1)$$
$$= \pi\lambda^2\tilde{v}^2(\omega_1) = \gamma, \quad (147)$$

$$\Gamma(t)h(t) \simeq \int_0^\infty d\omega\, I(\omega)\coth(\beta\hbar\omega/2)\frac{\pi}{2}\delta(\omega-\omega_1)$$
$$= \frac{\pi}{2}I(\omega_1)\coth(\beta\hbar\omega_1/2) = \omega_1\gamma\coth(\beta\hbar\omega_1/2), \quad (148)$$

$$\Gamma(t)f(t) \simeq -\lambda^2\int_0^\infty d\omega\, I(\omega)\coth(\beta\hbar\omega/2)\mathcal{P}\frac{1}{\omega^2-\omega_1^2}$$
$$= -2\omega_1\mathcal{P}\int_0^\infty dk\frac{\lambda^2 v_k^2}{\omega_k^2-\omega_1^2}\coth(\beta\hbar\omega_k/2). \quad (149)$$

Upon comparison of the $\lambda^2 t$ approximation of the HPZ equation with the $\lambda^2$ approximation of our exact Markovian kinetic equation (136), we find that they are identical.[5]

Beyond the weak coupling limit, our equation gives the Markovian dynamics of the quantum Brownian oscillator valid even for strong coupling, and also valid for any time scale. For $t\to\infty$ the solution of our equation gives the equilibrium solution of the complete dynamics. This is so because the equilibrium distribution is a function of the Hamiltonian, and any function of the Hamiltonian belongs to the $\Pi$ subspace [see Eq. (96)]. The complement component $\hat{\Pi}\rho$ gives all the memory effects, which vanish for $t\to\infty$.

## IX. CONCLUDING REMARKS

The example presented in this paper shows that irreversible Markovian dynamics can be regarded as an exact dynamics taking place in the subspace of density operators $\Pi\rho$, for nonintegrable systems in the sense of Poincaré. The breaking of time symmetry in the equation

$$i\frac{\partial}{\partial t}\Pi\rho = L_H\Pi\rho \quad (150)$$

is due to $\Pi$ being non-Hermitian, and appears before we take the trace over the field. Thus from our point of view of irreversibility, rather than a consequence of coarse graining, is a property of the invariant subspaces of the Liouvillian. Time symmetry breaking appears because the construction of $\Pi$ involves generalized creation and annihilation operators $(A_1,\tilde{A}_1)$. These are eigenoperators of the Liouvillian with complex eigenvalues $(z_1,z_1^*)$ in either the lower or upper half plane. In this formulation we add no extra dissipative terms to the Liouvillian.

The formulation presented in this paper links stochastic processes and dynamics in a direct way. Once we have a Markovian kinetic equation we have a stochastic process described by Langevin-type equations without any memory terms. An interesting question is to see what is the spectrum of quantum noise associated with such Langevin equations (see also Ref. [28]).

In this paper we focused on the $\Pi\rho$ component of the density matrix. In a sense, this component corresponds to traditional thermodynamics. From the Markovian kinetic equation we can derive a nonequilibrium entropy and the second law of thermodynamics even for strong coupling. This could be considered in a subsequent publication.

In contrast to $\Pi\rho$, the complement component $\hat{\Pi}\rho$ gives "nontraditional" thermodynamics including memory effects. Deviations from thermodynamics in small quantum systems have been reported in Ref. [29]. It would be interesting to see how the behavior of $\hat{\Pi}\rho$ is related to these deviations, and what type of non-Markovian equation is obtained for this component.

[5]In Ref. [11] the operators $P^{(\nu_1)}$ were used to construct the subdynamics. As a result, the "anomalous" diffusion term in the HPZ equation did not appear in the kinetic equation derived in Ref. [11], because this term belongs to the "nonprivileged" components. In order to obtain this term one has to include nonprivileged components. In contrast, in the present paper we use the operator $P=\Sigma_{\nu_1}P^{(\nu_1)}$ and the anomalous diffusion term is included in the privileged components. A detailed calculation shows that both the HPZ equation and our equation are consistent with Eq. (136) of Ref. [11] in the one-particle sector. The anomalous diffusion term involves higher particle sectors, which were not considered in Ref. [11].

The model we considered is exactly solvable. For systems with nonlinear interactions we have to use a perturbative approach. It is our hope that some of the ideas presented in this paper will be useful for these systems.

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## APPENDIX A: COEFFICIENTS OF TRANSFORMED OPERATORS

For the integrable case we have

$$\bar{c}_{11} = -\bar{N}\frac{\bar{\omega}_1 + \omega_1}{2\omega_1}, \tag{A1}$$

$$\bar{c}_{1k} = \bar{N}\frac{\lambda V_k}{\omega_k - \bar{\omega}_1}, \tag{A2}$$

$$\bar{d}_{11} = -\bar{N}\frac{\bar{\omega}_1 - \omega_1}{2\omega_1}, \tag{A3}$$

$$\bar{d}_{1k} = -\bar{N}\frac{\lambda V_k}{\omega_k + \bar{\omega}_1}, \tag{A4}$$

$$\bar{c}_{kk} = 1, \tag{A5}$$

$$\bar{c}_{k1} = -\lambda V_k G^{+}(\omega_k)(\omega_k + \omega_1), \tag{A6}$$

$$\bar{d}_{k1} = -\lambda V_k G^{+}(\omega_k)(\omega_k - \omega_1), \tag{A7}$$

$$\bar{c}_{kk'} = 2\omega_1 \lambda V_k G^{+}(\omega_k)\frac{\lambda V_{k'}}{\omega_{k'} - \omega_k - i\epsilon} \quad (k \neq k'), \tag{A8}$$

$$\bar{d}_{kk'} = -2\omega_1 \lambda V_k G^{+}(\omega_k)\frac{\lambda V_{k'}}{\omega_{k'} + \omega_k}. \tag{A9}$$

The normalization constant $\bar{N}$ given by

$$\bar{N}^2 = \frac{\omega_1}{\bar{\omega}_1}\left(1 + \int_0^\infty dk \frac{4\omega_1\omega_k\lambda^2 v_k^2}{(\omega_k^2 - \bar{\omega}_1^2)^2}\right)^{-1}. \tag{A10}$$

For the nonintegrable case we have [15]

$$c_{11} = -N^{*}\frac{z_1^{*} + \omega_1}{2\omega_1}, \tag{A11}$$

$$c_{1k} = N^{*}\frac{\lambda V_k}{(\omega_k - z_1^{*})_{-}}, \tag{A12}$$

$$d_{11} = -N^{*}\frac{z_1^{*} - \omega_1}{2\omega_1}, \tag{A13}$$

$$d_{1k} = -N^{*}\frac{\lambda V_k}{\omega_k + z_1^{*}}, \tag{A14}$$

$$c_{k1} = -\lambda V_k G_d^{+}(\omega_k)(\omega_k + \omega_1), \tag{A15}$$

$$d_{k1} = -\lambda V_k G_d^{+}(\omega_k)(\omega_k - \omega_1), \tag{A16}$$

$$c_{kk'} = 2\omega_1 \lambda V_k G_d^{+}(\omega_k)\frac{\lambda V_{k'}}{\omega_{k'} - \omega_k - i\epsilon} \quad (k \neq k'), \tag{A17}$$

$$d_{kk'} = -2\omega_1 \lambda V_k G_d^{+}(\omega_k)\frac{\lambda V_{k'}}{\omega_{k'} + \omega_k}, \tag{A18}$$

and

$$\tilde{c}_{k1} = \bar{c}_{k1}, \quad \tilde{c}_{kk'} = \bar{c}_{kk'}, \tag{A19}$$

$$\tilde{d}_{k1} = \bar{d}_{k1}, \quad \tilde{d}_{kk'} = \bar{d}_{kk'}. \tag{A20}$$

We define $G_d^{+}(\omega_k)$ as

$$G_d^{+}(\omega_k) \equiv G^{+}(\omega_k) - i\frac{\pi N^2}{\omega_1}\delta_c(\omega_k - z_1), \tag{A21}$$

where $-N^2/(2\omega_1)$ is the residue of $G^{+}(\omega)$ at the pole $z_1$ in the second sheet. The normalization constant $N$ is

$$N^2 = \frac{\omega_1}{z_1}\left(1 + \int_0^\infty dk \frac{4\omega_1\omega_k\lambda^2 v_k^2}{(\omega_k^2 - z_1^2)_{+}^2}\right)^{-1}. \tag{A22}$$

These coefficients give the out eigenoperators of the Liouvillian, $\tilde{A}_k^{\dagger}$, $\tilde{A}_1^{\dagger}$, and their Hermitian conjugates. In previous publications (e.g., Ref. [12]) the in states were used to obtain decaying states for $t > 0$ in the Schrödinger picture. In this paper we consider observables in the Heisenberg picture, where the out operators are the ones that decay for $t > 0$.

## APPENDIX B: COMMUTATION RELATION

In this appendix we prove the commutation relation

$$[\tilde{A}_1, \tilde{A}_1^{\dagger}] = 0. \tag{B1}$$

Using the explicit forms of $\tilde{A}_1$, $\tilde{A}_1^{\dagger}$ we obtain

$$[\tilde{A}_1, \tilde{A}_1^{\dagger}] = |N|^2\left(\sum_{k=0}^{\infty}|c_{1k}|^2 - \sum_{k=0}^{\infty}|d_{1k}|^2 + \frac{\tilde{\omega}_1}{\omega_1}\right). \tag{B2}$$

We will show that the expression inside parentheses vanishes,

$$\sum_k |c_{1k}|^2 - \sum_k |d_{1k}|^2 + \frac{\widetilde{\omega}_1}{\omega_1} = 0. \quad \text{(B3)}$$

We know that $z_1 = \widetilde{\omega}_1 - i\gamma$ is the pole of $G^+(\omega)$, defined in Eq. (74). Therefore

$$\omega_1^2 - z_1^2 = \sum_{k=0}^{\infty} \frac{4\omega_1\omega_k\lambda^2 V_k^2}{(\omega_k^2 - z_1^2)_+} = \sum_{k=0}^{\infty} \frac{2\omega_1\lambda^2 V_k^2}{(\omega_k - z_1)_+} + \sum_{k=0}^{\infty} \frac{2\omega_1\lambda^2 V_k^2}{\omega_k + z_1}. \quad \text{(B4)}$$

Subtracting the complex conjugate expression and dividing the result by $2\omega_1(z_1 - z_1^*)$ we have

$$\sum_{k=0}^{\infty} \frac{\lambda^2 V_k^2}{|\omega_k - z_1|_+^2} - \sum_{k=0}^{\infty} \frac{\lambda^2 V_k^2}{|\omega_k + z_1|^2} = -\frac{z_1 + z_1^*}{2\omega_1} = -\frac{\widetilde{\omega}_1}{\omega_1}, \quad \text{(B5)}$$

which is equivalent to Eq. (B3). This proves the desired expression (B1).

From Eqs. (82) and (86) we find that $[a_1, \widetilde{B}_1^\dagger] = [\Sigma_k D_k, \widetilde{B}_1^\dagger]$. We then deduce the following other commutation relations

$$[\widetilde{A}_k, A_1^\dagger] = 2\pi i\lambda V_k(\omega_k - z_1)\,\delta_c(\omega_k - z_1),$$

$$[A_k, \widetilde{A}_1^\dagger] = -2\pi i\lambda V_k \frac{G^+(\omega_k)}{G^-(\omega_k)}(\omega_k - z_1^*)\,\delta_c(\omega_k - z_1^*). \quad \text{(B6)}$$

If the test functions contain singularities at $\omega_k = z_1$ or $\omega_k = z_1^*$, then these commutators are nonvanishing.

## APPENDIX C: PROOF OF ANALYTICITY

In this section we verify condition (C) on the $\Pi$ projector for the nonintegrable case. This means that

$$\lim_{\lambda\to 0} \Pi^\dagger(a_1^{\dagger m+1} a_1^n) = P(a_1^{\dagger m+1} a_1^n) = a_1^{\dagger m+1} a_1^n, \quad \text{(C1)}$$

$$\lim_{\lambda\to 0} \Pi^\dagger(a_1^{\dagger m} a_1^{n+1}) = P(a_1^{\dagger m} a_1^{n+1}) = a_1^{\dagger m} a_1^{n+1}, \quad \text{(C2)}$$

for all $m$, $n \geqslant 0$. Furthermore, $\Pi$ has to be expandable in a power series of $\lambda$. If $\Pi$ satisfies these conditions we will say, in short, that it is analytic at $\lambda = 0$. This property is not trivial, because in Eq. (94) there appear nonanalytic terms in the products or commutators of renormalized operators, as in Eq. (82) (see also Ref. [12]).

We have, using Eqs. (77) and (86),

$$\Pi^\dagger a_1^\dagger = \widetilde{B}_1^\dagger, \quad \Pi^\dagger a_1 = \widetilde{B}_1. \quad \text{(C3)}$$

Both expressions are analytic at $\lambda = 0$. Assuming that $\Pi^\dagger(a_1^{\dagger m} a_1^n)$, $\Pi^\dagger(a_1^{\dagger m-1} a_1^n)$, and $\Pi^\dagger(a_1^{\dagger m} a_1^{n-1})$ are analytic at $\lambda = 0$, we will show that the recursive expression (98) is also analytic. This will prove Eq. (C1) by recursion.

We start with the first term in the right hand side of Eq. (98),

$$\Pi^\dagger a_1^\dagger \cdot \Pi^\dagger(a_1^{\dagger m} a_1^n) = \widetilde{B}_1^\dagger \cdot \Pi^\dagger\left[\left(\widetilde{B}_1^\dagger + \sum_k \widetilde{D}_k^\dagger\right)^m \left(\widetilde{B}_1 + \sum_k \widetilde{D}_k\right)^n\right]. \quad \text{(C4)}$$

As we show now, this product generates nonanalytic terms, even if its two factors are analytic. In the products between $\widetilde{B}_1^\dagger$ outside the brackets and either $\widetilde{B}_1$ or $\widetilde{B}_1^\dagger$ inside the brackets there appears the term $\widetilde{A}_1^\dagger \widetilde{A}_1$, which has the following $\lambda$ dependence:

$$\widetilde{A}_1^\dagger \widetilde{A}_1 = g_0(\lambda) a_1^\dagger a_1 + \lambda \int dk\, g_1(k,\lambda) a_1^\dagger a_k + \cdots + \lambda^2 \int dk\, g_2(k,\lambda) a_k^\dagger a_k + \cdots. \quad \text{(C5)}$$

For the perturbation expansion to exist, the functions $g_0(\lambda)$, $g_1(k,\lambda)$, $g_2(k,\lambda)$, … must be analytic at $\lambda = 0$, with $g_0(0) = 1$. However, $g_2(k,\lambda)$ is not analytic at $\lambda = 0$. We have

$$\lambda^2 g_2(k,\lambda) = |N|^2 \lambda^2 v_k^2 \frac{1}{(\omega_k - z_1)_+} \frac{1}{(\omega_k - z_1^*)_-} = |N|^2 \frac{\lambda^2 v_k^2}{z_1 - z_1^*}\left(\frac{1}{(\omega_k - z_1)_+} - \frac{1}{(\omega_k - z_1^*)_-}\right). \quad \text{(C6)}$$

For $\lambda \to 0$ the term inside large parentheses goes to

$$\frac{1}{\omega_k - \omega_1 - i\epsilon} - \frac{1}{\omega_k - \omega_1 + i\epsilon} = 2\pi i\,\delta(\omega_k - \omega_1). \quad \text{(C7)}$$

Moreover we have [see Eq. (D6)]

$$z_1 - z_1^* = -2i\gamma = -2\pi i \lambda^2 \tilde{v}^2(\omega_1) + O(\lambda^3) \quad \text{(C8)}$$

with

$$\tilde{v}^2(\omega_k) \equiv v_k^2 \frac{dk}{d\omega_k}. \quad \text{(C9)}$$

This leads to

$$\lim_{\lambda\to 0} \lambda^2 g_2(k,\lambda) = -\frac{d\omega_k}{dk}\,\delta(\omega_k - \omega_1) \quad \text{(C10)}$$

which is nonzero.

Coming back to Eq. (C4), the term $\widetilde{B}_1^\dagger$ outside brackets in the right hand side can pair with either $m$ of the $\widetilde{B}_1^\dagger$ or $n$ of the $\widetilde{B}_1$ inside brackets. Thus all the nonanalytic terms involving $\widetilde{A}_1^\dagger \widetilde{A}_1$ are included in

$$[\Pi^\dagger a_1^\dagger \cdot \Pi^\dagger(a_1^{\dagger m} a_1^n)]_{\text{non}} = n(\widetilde{B}_1^\dagger \widetilde{B}_1)_{\text{non}} \cdot \Pi^\dagger(a_1^{\dagger m} a_1^{n-1}) + m(\widetilde{B}_1^{\dagger 2})_{\text{non}} \cdot \Pi^\dagger(a_1^{\dagger m-1} a_1^n). \quad \text{(C11)}$$

In order for Eq. (98) to be analytic, the second and third terms in the right hand side of Eq. (98) should cancel the nonanalytic terms of Eq. (C11). Combining Eq. (98) and Eq. (C11) we obtain

$$[\Pi^\dagger(a_1^{\dagger m+1}a_1^n)]_{\rm non} = n(\tilde{B}_1^\dagger\tilde{B}_1 + Y)_{\rm non}\cdot\Pi^\dagger(a_1^{\dagger m}a_1^{n-1}) + m(\tilde{B}_1^{\dagger 2} + X)_{\rm non}\cdot\Pi^\dagger(a_1^{\dagger m-1}a_1^n). \tag{C12}$$

From Eq. (C10) we get

$$\lim_{\lambda\to 0}(\tilde{B}_1^\dagger\tilde{B}_1)_{\rm non} = -\int_0^\infty d\omega_k \delta(\omega_k-\omega_1)a_k^\dagger a_k \tag{C13}$$

where we used $\lim_{\lambda\to 0}|c_{11}|^2=1$ and $\lim_{\lambda\to 0}|d_{11}|^2=0$. On the other hand we have

$$Y_{\rm non} = \sum_k |\tilde{c}_{k1}|^2 a_k^\dagger a_k = \int_0^\infty dk\, \lambda^2 v_k^2 |G^+(\omega_k)|^2(\omega_k+\omega_1)^2 a_k^\dagger a_k \tag{C14}$$

where we replace $\tilde{A}_k^\dagger\tilde{A}_k$ by $a_k^\dagger a_k$ in the thermodynamic limit [see Eq. (68)].

In the limit $\lambda\to 0$ we obtain [see Eq. (E9)]

$$\lim_{\lambda\to 0} Y_{\rm non} = \int_0^\infty d\omega_k \frac{1}{4\omega_1^2}\delta(\omega_k-\omega_1)(\omega_k+\omega_1)^2 a_k^\dagger a_k = \int_0^\infty d\omega_k \delta(\omega_k-\omega_1)a_k^\dagger a_k. \tag{C15}$$

Thus we get

$$\lim_{\lambda\to 0}(\tilde{B}_1^\dagger\tilde{B}_1 + Y)_{\rm non} = 0 \tag{C16}$$

which shows that the nonanalytic terms cancel.

Similarly one can show that

$$\lim_{\lambda\to 0}(\tilde{B}_1^{\dagger 2} + X)_{\rm non} = 0. \tag{C17}$$

Since we assume that $\Pi^\dagger(a_1^{\dagger m-1}a_1^n)$ and $\Pi^\dagger(a_1^{\dagger m}a_1^{n-1})$ are analytic, we conclude that Eq. (98) is analytic at $\lambda=0$. Thus, by recursion $\Pi^\dagger(a_1^{\dagger m+1}a_1^n)$ is analytic for arbitrary $m$, $n$.

We can show in the same way that the recursive expression (99) is analytic at $\lambda=0$, which proves Eq. (C2).

## APPENDIX D: EVALUATING $\tilde{\omega}_1$ AND $\lambda$ TO $O(\lambda^2)$

From Eq. (B4) we have

$$\omega_1^2 - z_1^2 - \int_0^\infty dk \frac{4\omega_1\omega_k\lambda^2 v_k^2}{(\omega_k^2 - z_1^2)_+} = 0. \tag{D1}$$

Approximating $z_1^2 \simeq \omega_1^2 + i\epsilon$ in the denominator we have

$$\omega_1^2 - \tilde{\omega}_1^2 + 2i\tilde{\omega}_1\gamma - \int_0^\infty dk \frac{4\omega_1\omega_k\lambda^2 v_k^2}{\omega_k^2 - \omega_1^2 - i\epsilon} = 0. \tag{D2}$$

Writing

$$\frac{1}{\omega_k^2-\omega_1^2-i\epsilon} = \mathcal{P}\frac{1}{\omega_k^2-\omega_1^2} + i\pi\delta(\omega_k^2-\omega_1^2) = \mathcal{P}\frac{1}{\omega_k^2-\omega_1^2} + \frac{i\pi}{2\omega_1}\delta(\omega_k-\omega_1), \tag{D3}$$

we then obtain for the real part and imaginary parts of (D2)

$$\tilde{\omega}_1^2 = \omega_1^2 - \mathcal{P}\int_0^\infty dk \frac{4\omega_1\omega_k\lambda^2 v_k^2}{\omega_k^2-\omega_1^2} + O(\lambda^4). \tag{D4}$$

Therefore,

$$\tilde{\omega}_1 \simeq \omega_1 - \mathcal{P}\int_0^\infty dk \frac{2\omega_k\lambda^2 v_k^2}{\omega_k^2-\omega_1^2} \tag{D5}$$

and [see Eq. (C9)]

$$\gamma = \pi\lambda^2\tilde{v}^2(\omega_1) + O(\lambda^4). \tag{D6}$$

## APPENDIX E: GREEN'S FUNCTION IN THE WEAK COUPLING APPROXIMATION

In this appendix we find an approximation of Green's function $G^+$ valid for weak coupling. We start by expanding the inverse function around the pole $z_1$. Since this function depends on $\omega_k^2$, i.e.,

$$[G^+(\omega_k^2)]^{-1} = \omega_1^2 - \omega_k^2 - \int_0^\infty dk' \frac{4\omega_1\omega_{k'}\lambda^2 v_{k'}^2}{\omega_{k'}^2 - \omega_k^2 - i\epsilon}, \tag{E1}$$

we make an expansion in the variable $\omega_k^2$ around $z_1^2$,

$$G^+(\omega_k^2)^{-1} = G^+(z_1^2)^{-1} + (\omega_k^2 - z_1^2)[G^+(z_1^2)^{-1}]' + \frac{1}{2}(\omega_k^2-z_1^2)^2[G^+(z_1^2)^{-1}]'' + \cdots. \tag{E2}$$

We have $G_k^+(z_1^2)^{-1}=0$. The first derivative term is given by [with $N$ defined in Eq. (A22)]

$$\left.\frac{d[G^+(\omega_k^2)]^{-1}}{d\omega_k^2}\right|_{\omega_k^2=z_1^2} = -\left(1 + \int_0^\infty dk' \frac{4\omega_1\omega_{k'}\lambda^2 v_{k'}^2}{(\omega_{k'}^2 - z_1^2)^2}\right) = -\frac{\omega_1}{z_1}\frac{1}{N^2}. \tag{E3}$$

For weak coupling we may neglect the second and higher derivative terms. This gives

$$G^+(\omega_k) = -\frac{z_1}{\omega_1}\frac{N^2}{\omega_k^2 - z_1^2} + \text{higher derivatives}. \tag{E4}$$

Furthermore for weak coupling we have $N=1+O(\lambda^2)$, $z_1 = \omega_1 + O(\lambda^2)$. Thus we get

$$G^+(\omega_k) \simeq \frac{1}{(z_1-\omega_k)(z_1+\omega_k)} \simeq \frac{1}{(\omega_1-\omega_k-i\epsilon)(\omega_1+\omega_k)} = \frac{1}{\omega_1^2-\omega_k^2-i\epsilon} = -\mathcal{P}\frac{1}{\omega_k^2-\omega_1^2} + \frac{i\pi}{2\omega_1}\delta(\omega_k-\omega_1) \tag{E5}$$

and similarly

$$G^-(\omega_k) \simeq \frac{1}{\omega_1^2-\omega_k^2+i\epsilon} = -\mathcal{P}\frac{1}{\omega_k^2-\omega_1^2} - \frac{i\pi}{2\omega_1}\delta(\omega_k-\omega_1). \tag{E6}$$

Another useful formula follows from the exact relation

$$4\pi i\lambda^2\tilde{v}^2(\omega_k)\omega_1|G^+(\omega_k)|^2 = G^+(\omega_k) - G^-(\omega_k) \tag{E7}$$

where $\tilde{v}^2(\omega_k)\equiv v_k^2 dk/d\omega_k$. Using Eqs. (E5) and (E6), we find that

$$G^+(\omega_k) - G^-(\omega_k) \simeq \frac{i\pi}{\omega_1}\delta(\omega_k-\omega_1). \tag{E8}$$

Combining this result with Eq. (E7) we get

$$4\lambda^2\tilde{v}^2(\omega_k)|G^+(\omega_k)|^2 \simeq \frac{1}{\omega_1^2}\delta(\omega_k-\omega_1) \tag{E9}$$

in the lowest order approximation in λ expansion.

## APPENDIX F: TRACE RELATIONS INVOLVING $a_1^\dagger$ AND $a_1$

Using the relationships

$$[a_1, a_1^{\dagger n}] = n a_1^{\dagger n-1}, \tag{F1}$$

$$[a_1^\dagger, a_1^m] = -m a_1^{m-1}, \tag{F2}$$

$$[a_1^\dagger a_1, a_1^{\dagger n}] = n a_1^{\dagger n}, \tag{F3}$$

$$[a_1^\dagger a_1, a_1^m] = -m a_1^m, \tag{F4}$$

$$[a_1^{\dagger 2}, a_1^{\dagger n}a_1^m] = -2m a_1^{\dagger n+1}a_1^{m-1} - m(m-1)a_1^{\dagger n}a_1^{m-2}, \tag{F5}$$

$$[a_1^2, a_1^{\dagger n}a_1^m] = 2n a_1^{\dagger n-1}a_1^{m+1} + n(n-1)a_1^{\dagger n-2}a_1^m, \tag{F6}$$

we can show that

$$(n-m)\mathrm{Tr}(a_1^{\dagger n}a_1^m\tilde{\rho}) = \mathrm{Tr}([a_1^\dagger a_1, a_1^{\dagger n}a_1^m]\tilde{\rho}),$$

$$(n+m)\mathrm{Tr}(a_1^{\dagger n}a_1^m\tilde{\rho}) = \mathrm{Tr}(\{a_1^\dagger a_1, a_1^{\dagger n}a_1^m\}\tilde{\rho}) - 2\mathrm{Tr}(a_1^{\dagger n}a_1^m a_1\tilde{\rho}a_1^\dagger),$$

$$nm\mathrm{Tr}(a_1^{\dagger n-1}a_1^{m-1}\tilde{\rho}) = \mathrm{Tr}(a_1^{\dagger n}a_1^m a_1^\dagger\tilde{\rho}a_1) - (n+m+1)\mathrm{Tr}(a_1^{\dagger n}a_1^m\tilde{\rho}) - \mathrm{Tr}(a_1^{\dagger n}a_1^m a_1\tilde{\rho}a_1^\dagger),$$

$$\mathrm{Tr}(m a_1^{\dagger n+1}a_1^{m-1}\tilde{\rho}) = -\mathrm{Tr}(a_1^{\dagger n+1}[a_1^\dagger, a_1^m]\tilde{\rho}) = \mathrm{Tr}[a_1^{\dagger n}a_1^m(a_1^\dagger\tilde{\rho}a_1^\dagger - \tilde{\rho}a_1^\dagger a_1^\dagger)],$$

$$\mathrm{Tr}(n a_1^{\dagger n-1}a_1^{m+1}\tilde{\rho}) = \mathrm{Tr}([a_1^\dagger, a_1^n]a_1^{\dagger m+1}\tilde{\rho}) = \mathrm{Tr}[a_1^{\dagger n}a_1^m(a_1\tilde{\rho}a_1 - a_1 a_1\tilde{\rho})],$$

$$\mathrm{Tr}[m(m-1)a_1^{\dagger n}a_1^{m-2}\tilde{\rho}] = -\mathrm{Tr}(2m a_1^{\dagger n+1}a_1^{m-1}\tilde{\rho}) - \mathrm{Tr}([a_1^\dagger a_1^\dagger, a_1^{\dagger n}a_1^m]\tilde{\rho}) = -\mathrm{Tr}[a_1^{\dagger n}a_1^m(2a_1^\dagger\tilde{\rho}a_1^\dagger - a_1^\dagger a_1^\dagger\tilde{\rho} - \tilde{\rho}a_1^\dagger a_1^\dagger)],$$

$$\mathrm{Tr}[n(n-1)a_1^{\dagger n-2}a_1^m\tilde{\rho}] = -\mathrm{Tr}(2n a_1^{\dagger n-1}a_1^{m+1}\tilde{\rho}) + \mathrm{Tr}([a_1 a_1, a_1^{\dagger n}a_1^m]\tilde{\rho}) = -\mathrm{Tr}[a_1^{\dagger n}a_1^m(2a_1\tilde{\rho}a_1 - a_1 a_1\tilde{\rho} - \tilde{\rho}a_1 a_1)]. \tag{F7}$$

Furthermore,

$$a_1^\dagger a_1\tilde{\rho} + \tilde{\rho}a_1^\dagger a_1 - a_1\tilde{\rho}a_1^\dagger - a_1^\dagger\tilde{\rho}a_1 + \tilde{\rho} = \frac{1}{2}(a_1^\dagger a_1\tilde{\rho} + \tilde{\rho}a_1^\dagger a_1 - 2a_1\tilde{\rho}a_1^\dagger + a_1 a_1^\dagger\tilde{\rho} + \tilde{\rho}a_1 a_1^\dagger - 2a_1^\dagger\tilde{\rho}a_1). \tag{F8}$$

## APPENDIX G: COORDINATE REPRESENTATION OF $a_1^\dagger$ AND $a_1$

Starting from

$$a_1^\dagger = \sqrt{\frac{M_1\omega_1}{2}}\left(q_1 - i\frac{p_1}{M_1\omega_1}\right) \quad \text{and} \quad p_1 = \frac{1}{i}\frac{\partial}{\partial q_1}, \tag{G1}$$

for an arbitrary vector $|\phi\rangle$, we find that

$$\langle x_1|a_1^\dagger|\phi\rangle = \frac{1}{\sqrt{2}}\left(x_1 - \frac{\partial}{\partial x_1}\right)\langle x_1|\phi\rangle, \tag{G2}$$

$$\langle x_1|a_1|\phi\rangle = \frac{1}{\sqrt{2}}\left(x_1 + \frac{\partial}{\partial x_1}\right)\langle x_1|\phi\rangle, \tag{G3}$$

$$\langle\phi|a_1^\dagger|x'_1\rangle = \frac{1}{\sqrt{2}}\left(x'_1 + \frac{\partial}{\partial x'_1}\right)\langle\phi|x'_1\rangle, \tag{G4}$$

$$\langle\phi|a_1|x'_1\rangle = \frac{1}{\sqrt{2}}\left(x'_1 - \frac{\partial}{\partial x'_1}\right)\langle\phi|x'_1\rangle. \tag{G5}$$

The ket of the dimensionless coordinate $x_1$ is related to $q_1$ by $|q_1\rangle = (M_1\omega_1)^{1/4}|x_1\rangle$. From the relation $a_1 a_1^\dagger - a_1^\dagger a_1 = 1$, we also find that

$$\langle x_1|a_1 a_1^\dagger|\phi\rangle = \frac{1}{2}\left(x_1 + \frac{\partial}{\partial x_1}\right)\left(x_1 - \frac{\partial}{\partial x_1}\right)\langle x_1|\phi\rangle,$$

$$\langle x_1|a_1^\dagger a_1|\phi\rangle = \frac{1}{2}\left(x_1 - \frac{\partial}{\partial x_1}\right)\left(x_1 + \frac{\partial}{\partial x_1}\right)\langle x_1|\phi\rangle,$$

$$\langle\phi|a_1 a_1^\dagger|x'_1\rangle = \frac{1}{2}\left(x'_1 + \frac{\partial}{\partial x'_1}\right)\left(x'_1 - \frac{\partial}{\partial x'_1}\right)\langle\phi|x'_1\rangle,$$

$$\langle\phi|a_1^\dagger a_1|x'_1\rangle = \frac{1}{2}\left(x'_1 - \frac{\partial}{\partial x'_1}\right)\left(x'_1 + \frac{\partial}{\partial x'_1}\right)\langle\phi|x'_1\rangle. \tag{G6}$$

We then deduce that

$$\langle x_1|[a_1^\dagger a_1,\tilde{\rho}]|x'_1\rangle = \frac{1}{2}\left[-\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x'^2_1}\right)+(x_1^2-x'^2_1)\right]\tilde{\rho}(x_1,x_1'),$$

$$\langle x_1|([a_1\tilde{\rho},a_1^\dagger]+[a_1,\tilde{\rho}a_1^\dagger]+[a_1^\dagger\tilde{\rho},a_1]+[a_1^\dagger,\tilde{\rho}a_1])|x'_1\rangle = \left[-(x_1-x'_1)^2+\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x'_1}\right)^2\right]\tilde{\rho}(x_1,x'_1),$$

$$\langle x_1|([a_1\tilde{\rho},a_1^\dagger]+[a_1,\tilde{\rho}a_1^\dagger]-[a_1^\dagger\tilde{\rho},a_1]-[a_1^\dagger,\tilde{\rho}a_1])|x'_1\rangle = \left[-(x_1-x'_1)\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x'_1}\right)+\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x'_1}\right)(x_1+x'_1)\right]\tilde{\rho}(x_1,x_1'),$$

$$\langle x_1|([a_1^\dagger\tilde{\rho},a_1^\dagger]+[a_1^\dagger,\tilde{\rho}a_1^\dagger]+[a_1\tilde{\rho},a_1]+[a_1,\tilde{\rho}a_1])|x'_1\rangle = -\left[(x_1-x'_1)^2+\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x'_1}\right)^2\right]\tilde{\rho}(x_1,x_1'),$$

$$\langle x_1|([a_1^\dagger\tilde{\rho},a_1^\dagger]+[a_1^\dagger,\tilde{\rho}a_1^\dagger]-[a_1\tilde{\rho},a_1]-[a_1,\tilde{\rho}a_1])|x'_1\rangle = 2(x_1-x'_1)\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x'_1}\right)\tilde{\rho}(x_1,x_1'),$$

$$\langle x_1|([a_1^\dagger,\tilde{\rho}a_1^\dagger]+[a_1\tilde{\rho},a_1])|x'_1\rangle = -\frac{1}{2}\left[(x_1-x'_1)^2+\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x'_1}\right)^2+(x_1-x'_1)\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x'_1}\right)+\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x'_1}\right)(x_1+x'_1)\right]\tilde{\rho}(x_1,x_1'),$$

$$\langle x_1|([a_1^\dagger,\tilde{\rho}a_1^\dagger]-[a_1\tilde{\rho},a_1])|x'_1\rangle = \frac{1}{2}\left[(x_1^2-x'^2_1)+\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x'^2_1}\right)+2(x_1-x'_1)\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x'_1}\right)\right]\tilde{\rho}(x_1,x_1'). \quad \text{(G7)}$$


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